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Pseudoarcs, pseudocircles, Lakes of Wada and generic maps on S^2

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Abstract

We prove a Bruckner–Garg type theorem for the fiber structure of a generic map from a continuum X into the unit interval I. We also study the specific case of $X = S^2$. We show that each nondegenerate component of each fiber of a generic map in $C(S^2, I)$ is figure-eight-like. This together with a result by Krasinkiewicz and Levin gives that each nondegenerate component of each fiber of a generic map in $C(S^2, I)$ is hereditarily indecomposable and figure-eight-like. We also show that pseudoarcs, pseudocircles and Lakes of Wada appear in abundance in fibers of a generic map in $C(S^2, I)$. We also exhibit a general method for proving when a P-like hereditarily indecomposable continuum is Q-like when Q is a certain graph containing P.

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1. Introduction

The fiber structure of a generic continuous function on a closed interval has been extensively studied in real analysis by various authors. The classical Bruckner–Garg Theorem [1] describes the fiber structure of a generic continuous function. Morayne and the second author described the fiber structure of a generic smooth function in [6]. D'Aniello and the second author described the "worst case" behavior of the fiber structure of smooth functions in [5]. Kirchheim in [7] described the Hausdorff dimension of fibers of a generic map from $\mathbb{R}^n$ into $\mathbb{R}^m$. Further measure theoretic properties of generic continuous functions and monotone continuous functions defined on an interval were recently studied by the first author in [2,3]. In this paper, we study the fiber structure of a generic map from a continuum X into the unit interval I from a topological viewpoint.

Our first result shows that when viewed appropriately, the Bruckner–Garg Theorem holds in a very general setting. Let us first state the Bruckner–Garg Theorem. In the following, X is the unit interval.

Theorem 1.1. *A generic $f \in C(X, I)$ has the property that there is a countable dense set $D \subseteq (\min f, \max f)$ such that*

- *$f^{-1}(y)$ is a singleton set if $y \in \{\min f, \max f\}$;*
- *$f^{-1}(y)$ is homeomorphic to a Cantor set when $y \in (\min f, \max f) \setminus D$;*
- *$f^{-1}(y)$ is homeomorphic to the union of a Cantor set and an isolated point when $y \in D$.*

Theorem 3.10 shows that the above theorem holds for a nondegenerate continuum X when $f^{-1}(y)$ is replaced by $\mathrm{Comp}(f^{-1}(y))$, the space whose elements are components of $f^{-1}(y)$ and the topology is the upper semicontinuous topology. Moreover, the isolated points of $\mathrm{Comp}(f^{-1}(y))$ are actually degenerate continua where f attains local extrema. Since it is easy to show that a generic function from the interval into the interval is nowhere constant, the Bruckner–Garg Theorem is a special case of Theorem 3.10. Our proof of Theorem 3.10 makes an extensive use of the Boundary Bumping Theorem.

Krasinkiewicz [9] and Levin [10] independently showed that a generic map from a compactum X into I has the property that each of its fibers is a Bing compactum, a compactum all of whose components are hereditarily indecomposable. This gives us information about individual components of a generic map while Theorem 3.10 gives us information about the global structure of the fibers. As the case of $X = I$ is completely described, we investigate the higher dimensional phenomenon. To avoid unnecessary technical problems at the boundary and separation of cases, we work with $X = S^2$, the 2-sphere in $\mathbb{R}^3$, instead of $X = I^2$.

We know the global structure of fibers of a generic map $f \in C(S^2, I)$ and we know that each component of each fiber must be hereditarily indecomposable. However, there are many nonhomeomorphic hereditarily indecomposable continua on S^2. Our first step in understanding these components is to show that each such component must be either a point or figure-eight-like. As an intermediate step, we show using Krasinkiewicz's characterization of hereditarily indecomposable continua [8] that every hereditarily indecomposable continuum which is P-like is Q-like where Q is figure-eight and P is any nondegenerate subcontinuum of Q. The technique developed for this result is general and may be useful elsewhere.

We next show that the pseudoarc, and the pseudocircle occur naturally as the components of the fibers of a generic map $f \in \mathcal{C}(S^2, I)$. Indeed, we show that for a generic map $f \in \mathcal{C}(S^2, I)$, for almost all $y \in f(S^2)$, all components of $f^{-1}(y)$ are either points, pseudoarcs or pseudocircles. Perhaps, the more surprising fact is that the Lakes of Wada continuum appears as component. Recall that a Lakes of Wada continuum is a hereditarily indecomposable continuum M which separates S^2 into three pieces and is the boundary of each component of $S^2 \setminus M$. A "smooth" map from S^2 into I may have saddle points and these saddle points are stable under small perturbation. They translate into Lakes of Wada continua in the generic case. Indeed, for a generic map $f \in \mathcal{C}(S^2, I)$ there is a countable dense set $D \subseteq f(S^2)$ such that for each $y \in D$, there is a component of $f^{-1}(y)$ which is a Lakes of Wada continuum.

2. Definitions and background information

In this section we state background terminology and theorems, most of which can be found in [11,13].

Let X be a complete metric space. The boundary and the closure of a subset A will be denoted by ∂A and $\overline{A}$, respectively. A set $M \subseteq X$ is of first category if it is the countable union of nowhere dense sets. We say that a generic $x \in X$ satisfies property P means that $\{x \in X\colon\ x$ does not satisfy property $P\}$ is of first category in X. Often, particularly in real analysis, the word "typical" is used instead of generic. We will also say that $M \subseteq X$ is residual in X. This simply means $X \setminus M$ is of first category.

Let X be a Polish space, i.e., a complete, separable metric space, then $\mathcal{K}(X)$ denotes the set of all compact subsets of X with the Hausdorff metric, d_H. Recall that $\mathcal{K}(X)$ is Polish and if X is compact then so is $\mathcal{K}(X)$.

If X is a compact metric space, then $C(X, I)$ denotes the set of continuous functions from X into the unit interval I endowed with sup norm. We recall that $C(X, I)$ is a Polish space. A *map* is simply a continuous function. We use S^2 to denote the 2-sphere in $\mathbb{R}^3$. We point out $C(S^2, I)$ is homeomorphic to the set of all continuous functions $f : \mathbb{R}^2 \to I$ which have a limit at infinity.

A *continuum* is a compact connected metric space. A continuum is *degenerate* if it has only one point. Otherwise, we say that it is nondegenerate. The following fact about continua will be used frequently and can be found in Chapter V of [13].

Theorem 2.1 (Boundary Bumping Theorem). *Suppose that X is a continuum, U is open in X, $U \neq X$ and $p \in U$. Then, there is a connected set C containing p and contained in U whose closure intersects the boundary of U.*

We also use the following fact which follows easily from the Boundary Bumping Theorem. If U is any nonempty open subset of a nondegenerate continuum X, then U contains a nondegenerate continuum.

We say that a continuum P *is a graph* (see [13, Chapter IX]) if there are arcs $A_1, A_2, \ldots, A_n$ such that $P = \bigcup_{i=1}^{n} A_i$ and for $i \neq j$, $A_i \cap A_j$ has at most one point and this point is an endpoint of A_i as well as an endpoint of A_j. We call such $A_1, A_2, \ldots, A_n$ a *defining sequence* for the graph P. If $x \in P$ is an endpoint of some A_k and x belongs to no other A_i's, we call x *an endpoint of* P. We note that this is equivalent to the usual definition of a point being an endpoint of a continuum, i.e., a point where the continuum has order one. Again refer to Chapter IX of [13] for more details. If a and b are points of a given arc A then we will use the notation $[a, b]$ for the subarc determined by a and b.

A map f from a metric space X onto a metric space Y is an ε-map means that $\varepsilon > 0$ and $\operatorname{diam}(f^{-1}(y)) < \varepsilon$ for every $y \in Y$. We say that continuum X is P*-like* if for every $\varepsilon > 0$, there is an ε-map from X onto P. We are generally interested in continua which are P-like for some graph P. The reader may refer to Chapter II of [13] for basic facts about P-like continua. A continuum which is arc-like (circle-like) is often called *chainable* (*circularly chainable*). Figure-eight is a continuum homeomorphic to the union of two circles which intersect in exactly one point.

A continuum is *decomposable* if it is the union of two proper subcontinua. Otherwise we call it *indecomposable*. A continuum is *hereditarily indecomposable* if each of its subcontinua is indecomposable.

A subset C of a continuum X is a *composant* of X means that there is $p \in X$ such that $C = \bigcup\{K\colon K$ is a proper subcontinuum of X containing $p\}$. We recall that each composant of X is dense in X. We also recall that a nondegenerate continuum is indecomposable iff it has uncountably many pairwise disjoint composants. We refer the reader to Chapter XI of [13] for proofs of these statements and further information on composants.

A *pseudoarc* is a hereditarily indecomposable continuum which is arc-like. We recall that up to homeomorphism the pseudoarc is unique. For more information on the pseudoarc, the reader may refer to Lewis's comprehensive survey article [12]. A *pseudocircle* is a hereditarily indecomposable continuum which is circle-like. There are uncountably many nonhomeomorphic pseudocircles. However, in the plane or S^2, up to homeomorphism, there is only one pseudocircle. We again refer the reader to [12] for further information. We call a continuum $X \subseteq S^2$ *a Lakes of Wada continuum* if it is hereditarily indecomposable, $S^2 \setminus X$ has exactly three components and X is the boundary of each of these components.

We next recall the definition and the basic properties of upper semicontinuous decompositions of a space and the corresponding topology.

Suppose that X is a compact metric space and $\mathcal{D}$ is a decomposition of X into closed sets. For each open set U in X, we define

$$[U] = \{A \in \mathcal{D}\colon\ A \subseteq U\}.$$

If $\bigcup[U]$ is open in X for each open set $U \subseteq X$, then we say that $\mathcal{D}$ is an upper semicontinuous decomposition of X. The collection $\{[U]\colon U$ is open in $X\}$ is a basis for a topology on $\mathcal{D}$ and the corresponding topology will be referred to as the upper semicontinuous topology on X generated by $\mathcal{D}$. We remark that the condition in the definition of an upper semicontinuous decomposition is equivalent to saying that for each closed set F in X, the following set is closed

$$\bigcup\{A \in \mathcal{D}\colon\ A \cap F \neq \emptyset\}.$$

For notational convenience, we use $]F[$ to denote the set $\{A \in \mathcal{D}\colon\ A \cap F \neq \emptyset\}$. The following is a standard fact about upper semicontinuous topology.

Proposition 2.2 [13, Theorem 3.9]. *Suppose that X is a compact metric space and $\mathcal{D}$ is an upper semicontinuous decomposition of X. Then, $\mathcal{D}$ is a compact metric space.*

Throughout the rest of this section, we assume that X is a compact metric space. We use $\mathrm{Comp}(X)$ to denote the set of components of X.

Proposition 2.3. $\mathrm{Comp}(X)$ *is an upper semicontinuous decomposition of X and hence a compact metric space.*

Proof. Let F be a closed set in X and let p be a limit point of $\bigcup]F[$. Let $\{p_i\}$ be a sequence of points in $\bigcup]F[$ which converges to p and let $A_i \in]F[$ be such that $p_i \in A_i$. As each A_i is a continuum in the compact space X, some subsequence of $\{A_i\}$ converges in the Hausdorff metric to a continuum A. Since each $A_i \cap F \neq \emptyset$, we have that $A \cap F \neq \emptyset$. Since p is the limit of $\{p_i\}$, $p \in A$. Therefore, B, the component of X containing A, is in $]F[$. Hence, $p \in \bigcup]F[$ which, in turn, implies that $\bigcup]F[$ is closed and $\mathrm{Comp}(X)$ is upper semicontinuous. $\square$

Theorem 2.4. $\mathrm{Comp}(X)$ *is totally disconnected, i.e., the only components of* $\mathrm{Comp}(X)$ *are singletons.*

Proof. Let $\mathcal{M} \subseteq \mathrm{Comp}(X)$ be a continuum. It will suffice to show that $\bigcup \mathcal{M}$ is a continuum in X. It is easy to verify that $\bigcup \mathcal{M}$ is closed. To obtain a contradiction, assume that $\bigcup \mathcal{M}$ is not connected. Let H, K be two disjoint nonempty closed sets in X such that $\bigcup \mathcal{M} = H \cup K$. As each $A \in \mathcal{M}$ is connected, we have that $A \subseteq H$ or $A \subseteq K$. Let $\mathcal{M}_H = \{A \in \mathcal{M}\colon\ A \subseteq H\}$ and $\mathcal{M}_K = \{A \in \mathcal{M}\colon\ A \subseteq K\}$. Then, $\mathcal{M}_H$ and $\mathcal{M}_K$ are two disjoint nonempty closed sets in $\mathrm{Comp}(X)$ whose union is $\mathcal{M}$, contradicting that $\mathcal{M}$ is connected. $\square$

We make a frequent use of the Tietze Extension Theorem. We state it below for a compact metric space, the context in which it is used.

Theorem 2.5 (Tietze Extension Theorem [11]). *Suppose that X is a compact metric space, M is a closed subset of X and $f : M \to \mathbb{R}$ is a continuous function. Then, there is a continuous extension g of f to X such that the range of g is a subset of* $[\min f, \max f]$.

3. Bruckner–Garg Theorem for continua

In this section we prove the Bruckner–Garg Theorem for continua. The following result is well-known and simply follows from the fact that X is separable. Throughout this section, X is at least a compact metric space. After the next several results, X is assumed to be a nondegenerate continuum.

Theorem 3.1. *For each $f \in C(X, I)$, there are at most countably many values in I where f attains local extrema.*

Lemma 3.2. *Let $U = (u_1, u_2) \subseteq I$, $f \in C(X, I)$, and $\varepsilon > 0$. Then, there is a $g \in C(X, I)$ and $\gamma > 0$ such that*

(1) $\|f - g\| < \varepsilon$,
(2) *either $g(X) \cap U = \emptyset$, or $g(X) \cap U$ contains a local maximum as well as a local minimum of g.*

Moreover, if $h \in C(X, I)$ and $\|g - h\| < \gamma$, then h satisfies the above properties as well.

Proof. Suppose that $f(X) \cap U = \emptyset$. Then, f can be slightly modified to a continuous function g so that $g(X) \cap \overline{U} = \emptyset$. Let $\gamma > 0$ be less than the distance which separates $g(X)$ and $\overline{U}$.

Let us now consider the case when $f(X) \cap U \neq \emptyset$. Let $p \in X$ be such that $f(p) \in U$. Let $\delta > 0$ be small enough so that $\operatorname{osc}(f, B_\delta(p)) < \frac{\varepsilon}{4}$ and $[a, b] \subseteq U$ where $a = \inf(f(B_\delta(p)))$ and $b = \sup(f(B_\delta(p)))$. Let $a - \frac{\varepsilon}{4} < r < a$ and $b < s < b + \frac{\varepsilon}{4}$ be such that $[r, s] \subseteq U$. If $B_\delta(p)$ contains only one point of X, then we let $g = f$ and $\gamma = \frac{1}{4} \cdot \min\{\varepsilon, |u_1 - r|, |u_2 - s|\}$ and we are done. So let us assume that $B_\delta(p)$ contains at least two points. Let $p_1, p_2 \in B_\delta(p)$ and $\tilde{g} : (X \setminus B_\delta(p)) \cup \{p_1, p_2\} \to I$ be defined by $\tilde{g}(x) = f(x)$ if $x \in X \setminus B_\delta(p)$, $\tilde{g}(p_1) = r$ and $\tilde{g}(p_2) = s$. Let g be a continuous extension of $\tilde{g}$ to g so that $g(B_\delta(p)) \subseteq [r, s]$. Note that $\|g - f\| < \frac{3\varepsilon}{4}$ and $g(p_1)$ and $g(p_2)$ are local minimum and local maximum values of g, respectively. Let $\gamma = \frac{1}{8} \min\{|r - a|, |s - b|, |r - u_1|, |s - u_2|, \varepsilon\}$. Then, g, γ are the desired objects. □

Theorem 3.3. *A generic $f \in C(X, I)$ has the property that its extreme values are dense in $f(X)$.*

Proof. For each open interval U_i with rational endpoints, let $\mathcal{G}_i = \{f \in C(X, I)$: either $f(X) \cap U_i = \emptyset$, or $f(X) \cap U_i$ contains an extreme value$\}$. By Lemma 3.2 we have that $\mathcal{G}_i$ contains a dense open subset of $C(X, I)$. Hence, $\mathcal{G} = \bigcap_{i=1}^{\infty} \mathcal{G}_i$ is the desired residual set. □

The following theorem follows in a fashion similar to the above.

Theorem 3.4. *A generic $f \in C(X, I)$ has the property that there is a dense subset of X (depending on f) where f attains a local extremum.*

Throughout the rest of this section, X is assumed to be a nondegenerate continuum in addition to being a compact metric space.

Lemma 3.5. *Suppose that $f \in C(X, I)$ and $\varepsilon > 0$. Then, there is a finite collection $\mathcal{U} = \mathcal{U}[f, \varepsilon]$ consisting of pairwise disjoint open balls in X, $g = g[f, \varepsilon] \in C(X, I)$ and $\gamma = \gamma[f, \varepsilon] > 0$ such that*

(1) *if $U \in \mathcal{U}$ then the diameter of U is less than ε and the boundary of U is nonempty,*
(2) $\|f - g\| < \varepsilon$,
(3) *if $p \in X$ then there are at least two points, p_1, p_2 in X such that if q equals one of them then $|p - q| < \varepsilon$ and $f(p) = g(q)$,*
(4) *if $p \in X \setminus \bigcup \mathcal{U}$, then there is q such that $0 < |p - q| < \varepsilon$ and $g(p) = g(q)$,*
(5) *if $p \in U \in \mathcal{U}$ and $g(p) \in [\min g(\partial(U)), \max g(\partial(U))]$, then there is $q \in X$ such that $0 < |p - q| < \varepsilon$ such that $g(p) = g(q)$, and*
(6) *if $U \in \mathcal{U}$, then the extrema of g relative to $\overline{U}$ are attained in the interior of U.*

Moreover, if $h \in C(X, I)$ and $\|h - g\| < \gamma$, then h satisfies the above properties as well.

Proof. As f is uniformly continuous, we may choose $0 < \delta < \frac{\varepsilon}{8}$ such that if $x, y \in X$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \frac{\varepsilon}{8}$. We may also assume that δ is less than the diameter of X which is greater than zero as X has at least two elements. Let $x_1, x_2, \ldots, x_n$ be finitely many points so that if $x \in X$, then there are at least two i's such that $|x - x_i| < \frac{\delta}{2}$. Now, choose open balls, U_i's, centered at x_i's with radii less than $\frac{\delta}{4}$ so that if $i \neq j$ then $U_i \cap U_j = \emptyset$. By the Boundary Bumping Theorem, we may choose a nondegenerate continuum $C_i \subseteq U_i$. Let us first define g on C_i so that $g(C_i) = [a_i - \frac{\varepsilon}{2}, b_i + \frac{\varepsilon}{2}]$ where $a_i = \min f(\overline{U}_i)$ and $b_i = \max f(\overline{U}_i)$. If $x \notin \bigcup_{i=1}^{n} U_i$, then let $g(x) = f(x)$. Using the Tietze Extension Theorem, extend g to all of X so that $g(U_i) = [a_i - \frac{\varepsilon}{2}, b_i + \frac{\varepsilon}{2}]$. Let $\mathcal{U} = \{U_i\colon i = 1, \ldots, n\}$, and $\gamma = \frac{\varepsilon}{8}$. We claim that $\mathcal{U}$, g, and γ are the desired objects. By the fashion in which δ was chosen and the fact that X is connected, we have that condition (1) is satisfied. Let $h \in C(X)$ with $\|g - h\| < \gamma$. By construction, $\|f - g\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{8}$. Therefore, $\|f - h\| \leqslant \|f - g\| + \|g - h\| < \frac{5\varepsilon}{8} + \frac{\varepsilon}{8} < \varepsilon$. Hence, condition (2) is satisfied for g and h. Let us proceed to condition (3). Let $p \in X$. Then, there are $i \neq j$ such that $|p - x_i| < \frac{\delta}{2}$ and $|p - x_j| < \frac{\delta}{2}$. We also have that $|f(p) - f(x_t)| < \frac{\varepsilon}{8}$ for $t = i, j$. As $f(p) \in [a_t - \frac{\varepsilon}{8}, b_t + \frac{\varepsilon}{8}] \subseteq [a_t - \frac{3\varepsilon}{8}, b_t + \frac{3\varepsilon}{8}] \subseteq h(U_t)$ for $t = i, j$, there is an element q in each of U_i and U_j such that $h(q) = f(p)$. Similarly, condition (4) follows from the fact that if $p \notin \bigcup \mathcal{U}$, then $h(p) \in [a_t - \frac{2\varepsilon}{8}, b_t + \frac{2\varepsilon}{8}]$ for a suitable t. To verify condition (5) assume that $p \in U_j$ for some j. There is $i \neq j$ such that $|p - x_i| < \frac{\delta}{2}$. Since the radius of U_j is less than $\frac{\delta}{4}$, for each r on the boundary of U_j we have that $|r - x_i| < \delta$ and hence $|f(r) - f(x_i)| < \frac{\varepsilon}{8}$. Since $f(r) = g(r)$ and $\gamma = \frac{\varepsilon}{8}$, we have that $h(r) \in [a_i - \frac{2\varepsilon}{8}, b_i + \frac{2\varepsilon}{8}] \subseteq [a_i - \frac{3\varepsilon}{8}, b_i + \frac{3\varepsilon}{8}] \subseteq h(U_i)$. To verify condition (6) we only need to observe that if p is on the boundary of U_i, then $h(p) \in [a_i - \frac{\varepsilon}{8}, b_i + \frac{\varepsilon}{8}] \subseteq [a_i - \frac{3\varepsilon}{8}, b_i + \frac{3\varepsilon}{8}] \subseteq h(U_i)$. □

Theorem 3.6. *A generic $g \in C(X, I)$ has the property that if $p \in X$ and p is an isolated point of $g^{-1}(g(p))$ then g has a local extremum at p.*

Proof. We have assumed that X is nondegenerate and hence it has at least two points. Let $\{f_i\}$ be a sequence dense in $C(X, I)$ and let $\varepsilon_{n,k} = \frac{1}{n+k}$. By Lemma 3.5 we may choose $\mathcal{U}_{n,k}$, $g_{n,k} = g[f_k, \varepsilon_{n,k}]$ and $\gamma_{n,k} = \gamma[f_k, \varepsilon_{n,k}]$ which satisfy the conclusion of the lemma with respect to f_k and $\varepsilon_{n,k}$. Let $\mathcal{B}_{n,k}$ be the ball in $C(X, I)$ centered at $g_{n,k}$ with radius $\gamma_{n,k}$. Then, $\mathcal{G}_n = \bigcup_{k=1}^{\infty} \mathcal{B}_{n,k}$ is dense and open in $C(X, I)$. Let $\mathcal{G} = \bigcap_{n=1}^{\infty} \mathcal{G}_n$. We show that $g \in \mathcal{G}$ satisfies the desired property. Let $\{j_i\}$ be a sequence so that $g \in \mathcal{B}_{i,j_i}$

for all i. Let $\mathcal{U}_i = \mathcal{U}[f_{j_i}, \varepsilon_{i,j_i}]$. Let $p \in X$. It will suffice to show that if g does not have a local extremum at p, then $g(p)$ is a limit point of $g^{-1}(g(p))$. Let us assume that g does not have a local extremum at p and let $\eta > 0$. Let n be large enough so that $\frac{1}{n} < \frac{\eta}{2}$. If $p \notin \bigcup \mathcal{U}_n$, then by condition (4) of Lemma 3.5 we know that there is a $q \in X$ with $0 < |p - q| < \varepsilon_{n,j_n} < \eta$ so that $g(q) = g(p)$. For the second case assume that $p \in \bigcup \mathcal{U}_n$ and $p \in U$ for some $U \in \mathcal{U}_n$. If $g(p) \in [\min g(\partial(U)), \max g(\partial(U))]$, then by condition (5) we have that there is q with $0 < |p - q| < \varepsilon_{n,j_n} < \eta$ and $g(p) = g(q)$. So let us assume that $g(p) \notin [\min g(\partial(U)), \max g(\partial(U))]$. Without loss of generality, we may assume that $g(p) > \max g(\partial(U))$. By condition (6) of Lemma 3.5 the maximum of g on $\overline{U}$ must occur in U. Let $q \in U$ where this maximum occurs. Then, $g(q) > g(p)$ as g does not have a local extremum at p. By the Boundary Bumping Theorem, we may obtain a continuum $C \subseteq \overline{U}$ containing q which intersects the boundary of U. We note that $\min g(C) < g(p) < \max g(C)$ as $q \in C$ and some point of the boundary of U is also in C. As $\{f_{j_i}\}_{i=1}^{\infty}$ converges uniformly to g on X, there exists $m > n$ such that $f_{j_m}(C)$ contains $g(p)$. Let $p' \in C$ be such that $f_{j_m}(p') = g(p)$. Since $g \in B_{m,j_m}$, we have by condition (3) of Lemma 3.5 that there are at least two q's within ε_{m,j_m} of p' so that $g(q) = f_{j_m}(p')$. Let q be one such point distinct from p. Then, $0 < |p - q| < |p - p'| + |p' - q| < \varepsilon_{n,j_n} + \varepsilon_{m,j_m} < \frac{\eta}{2} + \frac{\eta}{2} = \eta$ and $g(q) = f_{j_m}(p') = g(p)$. Hence, we have shown that if p is not a local extremum of g and $\eta > 0$ then, there is q with $0 < |p - q| < \eta$ with $g(p) = g(q)$, completing the proof of the theorem. □

Definition 3.7. Suppose that $f \in C(X, I)$ and $\varepsilon > 0$. We call the fiber $f^{-1}(y)$ *ε-fine* if for each $A \in \mathrm{Comp}(f^{-1}(y))$ with $\mathrm{diam}(A) \geqslant \varepsilon$, there is $B \in \mathrm{Comp}(f^{-1}(y))$ such that $\mathrm{diam}(B) < \varepsilon$, and points a, b in A, B, respectively, such that $d(a, b) < \varepsilon$. Note that if $0 < \varepsilon' \leqslant \varepsilon$ and a fiber is ε'-fine, then it is ε-fine.

Lemma 3.8. *Suppose that $f \in C(X, I)$ and $\varepsilon > 0$. Then, there is $g \in C(X, I)$ and $\gamma > 0$ such that*

(1) $\|f - g\| < \varepsilon$,
(2) *g is constant on no ball of radius ε,*
(3) *if $p_1, p_2 \in X$ with $d(p_1, p_2) \geqslant 3 \cdot \varepsilon$ and g restricted to $B_\varepsilon(p_i)$ has an extremum at p_i for $i = 1, 2$, then $g(p_1) \neq g(p_2)$, and*
(4) *each fiber of g is ε-fine.*

Moreover, if h is such that $\|h - g\| < \gamma$, then h satisfies the above properties.

Proof. Using the uniform continuity of f, choose $0 < \delta < \frac{\varepsilon}{8}$ so that if $x, y \in X$ with $d(x, y) < \delta$, then $|f(x) - f(y)| < \frac{\varepsilon}{8}$. Using the compactness of X, choose distinct points $x_1, x_2, \ldots, x_n$ in X so that if $x \in X$ then there is x_i such that $d(x, x_i) < \delta$. Let $U_1, U_2, \ldots, U_n$ be pairwise disjoint open balls of radii less than δ centered at $x_1, x_2, \ldots, x_n$, respectively. For each i, let V_i and W_i be balls centered at x_i such that $\overline{W}_i \subseteq V_i$, $\overline{V}_i \subseteq U_i$. As X is a nondegenerate continuum, by the Boundary Bumping Theorem we may choose a nondegenerate continuum $C_i \subseteq W_i$. Let $a_i = \min f(\overline{U}_i)$ and $b_i = \max f(\overline{U}_i)$. Let $\tilde{g} : [X \setminus (\bigcup_{i=1}^n U_i)] \cup \bigcup_{i=1}^n (C_i \cup (\overline{V}_i \setminus W_i)) \to \mathbb{R}$ be a continuous function such that

(i) $\tilde{g}(x) = f(x)$ if $x \in X \setminus (\bigcup_{i=1}^{n} U_i)$,
(ii) $\tilde{g}(C_i) = [r_i, s_i]$ where $[a_i - \frac{\varepsilon}{4}, b_i + \frac{\varepsilon}{4}] \subseteq (r_i, s_i) \subseteq [a_i - \frac{\varepsilon}{2}, b_i + \frac{\varepsilon}{2}]$,
(iii) $g(\overline{V}_i \setminus W_i) = r_i$, and
(iv) for all $i \neq j$, $r_i \neq r_j$, $s_i \neq s_j$, and for all i, j, $r_i \neq s_j$.

Now using the Tietze Extension Theorem, let g be an extension of $\tilde{g}$ such that $g(\overline{U}_i) \subseteq [r_i, s_i]$. Then, $|f(x) - g(x)| < \frac{5\varepsilon}{8}$ for all $x \in X$. Hence, condition (1) of the lemma is satisfied. That g is constant on no ball of radius ε follows from the fact that each such ball contains some C_i. Let us now verify condition (3). Suppose that $p_1, p_2 \in X$ with $d(p_1, p_2) \geqslant 3 \cdot \varepsilon$ and g restricted to $B_\varepsilon(p_i)$ has an extremum at p_i for each i. We first show that $p_i \in \bigcup_{k=1}^{n} U_k$. Suppose that $p \notin \bigcup_{k=1}^{n} U_k$. Then, there is k such that $d(p, x_k) < \delta$. This implies that $g(p) = f(p) \in [a_k - \frac{\varepsilon}{8}, b_k + \frac{\varepsilon}{8}]$. However, C_k is a subset of $B_\varepsilon(p)$ and $g(C_k) = [r_k, s_k]$ with $r_k < a_k - \frac{\varepsilon}{8}$ and $s_k > b_k + \frac{\varepsilon}{8}$. Hence, g restricted to $B_\varepsilon(p)$ does not have an extremum at p. Therefore, $p_1 \in U_{k_1}$ and $p_2 \in U_{k_2}$ for some $k_1 \neq k_2$. Since $g(p_i) \subseteq [r_{k_i}, s_{k_i}]$, g restricted to $B_\varepsilon(p_i)$ has a local extremum at p_i and $B_\varepsilon(p_i) \supseteq U_{k_i}$, we have that $g(p_i) \in \{r_i, s_i\}$. Hence from property iv it follows that $g(p_1) \neq g(p_2)$. Let us now show that each fiber of g is ε-fine. Let $y \in g(X)$ and $A \in \mathrm{Comp}(g^{-1}(y))$ with $\mathrm{diam}(A) \geqslant \varepsilon$. As $\{U_i\}$ is a pairwise disjoint sequence of open balls with diameter less than $\frac{\varepsilon}{8}$, there is $p \in A$ such that $p \in X \setminus \bigcup_{i=1}^{n} U_i$. Let k be such that $d(p, x_k) < \delta$. As before, we have that $g(p) \in [a_k - \frac{\varepsilon}{8}, b_k + \frac{\varepsilon}{8}]$. We know that there is $q \in C_k$ such that $g(q) = g(p) > r_k$. Then, B, the component of $g^{-1}(y)$ containing q is contained in W_k as $g(\overline{V}_k \setminus W_k) = r_k$. Now we have that B is the desired component as $d(p, q) < \varepsilon$ and $\mathrm{diam}(B) < \varepsilon$. Now let $\gamma > 0$ be less than

$$\frac{1}{8} \cdot \min\left\{\varepsilon, \left|r_i - \left(a_i - \frac{\varepsilon}{4}\right)\right|, \left|s_i - \left(b_i + \frac{\varepsilon}{4}\right)\right|, |r_i - r_j|, |s_i - s_j|, |r_i - s_i|, |r_i - s_j|\colon 1 \leqslant i, j \leqslant n \text{ and } i \neq j\right\}.$$

Then, γ is the desired constant. □

Theorem 3.9. *A generic $f \in C(X, I)$ satisfies the following conditions*:

(1) *Each fiber of f is nowhere dense.*
(2) *No fiber of f contains two points where the function has a local extremum.*
(3) *If $y \in f(X)$, and $A \in \mathrm{Comp}(f^{-1}(y))$ is an isolated point of $\mathrm{Comp}(f^{-1}(y))$ in the upper semicontinuous topology, then A is a singleton set.*

Proof. For each $\varepsilon > 0$, consider the collection $\mathcal{G}_\varepsilon$ of those functions $g \in C(X, I)$ which satisfy conditions (2)–(4) of Lemma 3.8. Then, $\mathcal{G}_\varepsilon$ contains a dense open set in $C(X, I)$. Let $\mathcal{G} = \bigcap_{n=1}^{\infty} \mathcal{G}_{\frac{1}{n}}$. Let $f \in \mathcal{G}$. It is clear that f satisfies conclusions (1) and (2) of the theorem. To see the last conclusion observe that each fiber of f is ε-fine for every $\varepsilon > 0$. □

Theorem 3.10. *For a generic $f \in C(X, I)$ there is a countable dense set $D \subseteq (\min f(X), \max f(X))$ such that*

(1) *if $y \in \{\min f(X), \max f(X)\}$, then $\mathrm{Comp}(f^{-1}(y))$ is a singleton set,*
(2) *if $y \in D$, then $\mathrm{Comp}(f^{-1}(y))$ is homeomorphic to the Cantor set union an isolated point, and*

(3) *if* $y \in (\min f(X), \max f(X)) \setminus D$*, then* $\mathrm{Comp}(f^{-1}(y))$ *is homeomorphic to the Cantor set.*

Proof. Let $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$, $\mathcal{G}_4$ be the set of $f \in C(X, I)$ which satisfy Theorems 3.1, 3.3, 3.6, 3.9, respectively. We want to show that if f is in the residual set $\mathcal{G} = \bigcap_{i=1}^{4} \mathcal{G}_i$, then f satisfies the theorem. If $y \in \{\min f(X), \max f(X)\}$, then by the second statement of Theorem 3.9 we have that $f^{-1}(y)$ has exactly one point and hence condition (1) of the theorem holds.

Let $D \subseteq f(X)$ be the set of extreme values of f different from $\{\min f, \max f\}$. By Theorems 3.1 and 3.3 we know that D is a countable set which is dense in $(\min f(X), \max f(X))$.

We next make an observation. If $A \in \mathrm{Comp}(f^{-1}(y))$ is an isolated point, then by the third statement of Theorem 3.9 we have that A is a singleton set. Letting $A = \{x\}$, we have that x is an isolated point of $f^{-1}(y)$ and hence by Theorem 3.6, f has a local extremum at x. By the second statement of Theorem 3.9 it follows that $\mathrm{Comp}(f^{-1}(y))$ has at most one isolated point. Hence, what we have just shown is that $\mathrm{Comp}(f^{-1}(y))$ is either perfect or has exactly one isolated point. The latter happens only at the extreme values of f. Conversely, if f has a local extremum at x, then by the second statement of Theorem 3.9, we have that x is an isolated point of $f^{-1}(y)$ and hence $\{x\}$ is an isolated point of $\mathrm{Comp}(f^{-1}(y))$. Combining these two pieces of information and utilizing Theorem 2.4, we have that if $y \in (\min f(X), \max f(X)) \setminus D$, then $\mathrm{Comp}(f^{-1}(y))$ is homeomorphic to the Cantor set. Utilizing Theorem 2.4 again, we know that for $y \in D$, $\mathrm{Comp}(f^{-1}(y))$ is either a singleton set or homeomorphic to the union of a set homeomorphic to the Cantor set and an isolated point. To finish the proof of the theorem, it suffices to show that $f^{-1}(y)$ has at least two components for each $y \in D$. By our observation above, it will suffice to show that $f^{-1}(y)$ has at least two points. Let $x \in X$ be such that $f(x) = y$ and say f has a local maximum at x. Let $x' \in X$ be such that $f(x') = \max f(X)$. We know that $x \neq x'$ as $f(x') > f(x)$. Let U be an open ball containing x such that $f(\overline{U}) \subseteq (-\infty, y]$. Then, $x' \in X \setminus \overline{U}$. By the Boundary Bumping Theorem, we may obtain a connected set C containing x', intersecting the boundary of U and contained in $X \setminus U$. Note that $f(C)$ contains y and hence we have another point which maps to y. □

4. When P-likeness implies Q-likeness

In this section we develop a general technique which can be used to show that a P-like hereditarily indecomposable continuum is Q-like when Q is a certain type of graph containing P. However, we first need some terminology.

Definition 4.1. Suppose that P is a graph and $x \in P$. We say that the graph Q is a *method I extension of* P, denoted by $P \prec_1 Q$, if $Q = P \cup A$ where A is an arc with x as one of its endpoints and $P \cap A = \{x\}$.

Definition 4.2. Suppose that P is a graph and b is an endpoint of P and $[a, b]$ is an arc in P such that $(a, b]$ is open in P. We say that graph Q is a *method II extension of* P, denoted by $P \prec_2 Q$, if $Q = P \cup A$ where A is an arc with endpoints a, b and $P \cap A = \{a, b\}$.

Definition 4.3. Suppose that P is a graph, b_1, b_2 are endpoints of P, and $[a, b_1], [a, b_2]$ are arcs in P such that $[a, b_1] \cap [a, b_2] = \{a\}$ and $(a, b_i]$ is open in P for $i = 1, 2$. We say that graph Q is a *method III extension of* P, denoted by $P \prec_3 Q$, if $Q = P \cup A$ where A is an arc with endpoints b_1, b_2 such that $A \cap P = \{b_1, b_2\}$.

Theorem 4.4. *Suppose that P is a graph, and M is a continuum which is P-like. If $P \prec_1 Q$, then M is Q-like.*

Proof. Let $Q = P \cup A$ where A is from the definition of $P \prec_1 Q$. Let $A_1, A_2, \ldots, A_n$ be a defining sequence for P. Without loss of generality, we may assume that $A \cap P$ is an endpoint of some A_t. Let $A_t = [a, b]$ and $b = A \cap P$. Let $\varepsilon > 0$. Let f be a map from M onto P and $\delta > 0$ so that if U is a set with diameter less than δ, then $f^{-1}(U)$ has diameter less than ε. Let $J = [a', b] \subseteq A_t$ be an arc with diameter less than δ. Now, let h be a continuous map from J onto $J \cup A$ so that $h(a') = a'$ and $h(b) = b$. Extend h to all of P by making it identity on $P \setminus J$. Then, h is a δ-map from P onto Q and $h \circ f$ is an ε-map from M onto Q. □

The following lemma was generalized in [4]. We include it here because the technique of the proof given below is used in Lemma 4.6.

Lemma 4.5. *Suppose that P is a graph, $\varepsilon > 0$, M is a continuum and $f : M \to P$ is an ε-map onto P. Then, there is $\eta > 0$ such that if N is a continuum and the Hausdorff distance from M to N is less than η, then there is a 2ε-map from N onto P.*

Proof. Using the continuity of f and the fact that P is a graph, we may choose a sequence of arcs $A_1, A_2, \ldots, A_n$ so that

(1) $\bigcup_{i=1}^{n} A_i = P$,
(2) if $x \in A_i \cap A_j$ with $i \neq j$, then x is an endpoint of A_i as well as A_j,
(3) if $\mathcal{F} \subseteq \{A_1, \ldots, A_n\}$ such that $\bigcup \mathcal{F}$ is a simple closed curve, then $\mathcal{F}$ has at least five elements, and
(4) the diameter of $f^{-1}(A_i)$ is less than ε for all i.

Now choose a sequence of open sets $V_1, V_2, \ldots, V_n$ in M such that for all $1 \leqslant i \leqslant j \leqslant n$,

- $f^{-1}(A_i) \subseteq V_i$,
- the diameter of $\overline{V}_i$ is less than ε,
- $\overline{V}_i \cap \overline{V}_j \neq \emptyset$ iff $A_i \cap A_j \neq \emptyset$, and
- $U_i = V_i \setminus (\bigcup_{j, j \neq i} \overline{V}_j) \neq \emptyset$ and $U_i \cap f^{-1}(A_i) \neq \emptyset$.

Such V_i's may be chosen in the following fashion. Let $\delta > 0$ be small enough so that for each i we have that if $f^{-1}(A_j) \cap f^{-1}(A_i) = \emptyset$, then $d(a,b) > \delta$ for all $a \in f^{-1}(A_i)$ and $b \in f^{-1}(A_j)$. For each i, let $x_i \in f^{-1}(A_i)$ be such that $f(x_i)$ is not an endpoint of A_i. Since $p \in A_i \cap A_j$ iff p is an endpoint of A_i and A_j, we can make δ small enough so that for all $a \in \bigcup_{j \neq i} f^{-1}(A_j)$, we have that $d(a, x_i) > \delta$, we can also assume that the same δ works for all i. Since the diameter of $f^{-1}(A_i)$ is less than ε, we can choose an open set V_i which contains $f^{-1}(A_i)$ such that the diameter of $\overline{V}_i$ is less than ε and $V_i \subseteq \bigcup_{a' \in f^{-1}(A_i)} B_{\frac{\delta}{2}}(a')$. Then, these V_i's have the desired properties.

Let $\eta > 0$ be small enough so that if N is a continuum within η of M in the Hausdorff metric, then $N \subseteq \bigcup_{i=1}^n V_i$ and $N \cap U_i \neq \emptyset$ for all i. In order to finish the proof, it will suffice to define a 2ε-map g from $\bigcup_{i=1}^n \overline{V}_i$ onto P such that $g(N) = P$.

Let us first make an observation. Let $x \in \overline{V}_i \cap \overline{V}_j$ for some $i \neq j$ and let $x \in \overline{V}_{i'} \cap \overline{V}_{j'}$ for some $i' \neq j'$. Then, $A_i \cap A_j = A_{i'} \cap A_{j'}$ and $A_i \cap A_j$ has exactly one element. That $A_i \cap A_j$ has exactly one element follows from properties (2) and (3) above. If $A_i \cap A_j \neq A_{i'} \cap A_{j'}$, then $A_i \cup A_j \cup A_{i'} \cup A_{j'}$ would contain a simple closed curve, contradicting property (3) above.

Let $x \in \overline{V}_i \cap \overline{V}_j$ for $i \neq j$. Define $g(x)$ to be the point common to $A_i \cap A_j$. By the observation above, g is well defined on $\bigcup_{(i,j), i \neq j} \overline{V}_i \cap \overline{V}_j$. From the properties of V_i's and the observation above, we also have that g is continuous. Since N is a continuum, by the Boundary Bumping Theorem, there is a nondegenerate continuum $N_i \subseteq N \cap U_i$. Define g continuously on N_i so that $g(N_i) = A_i$. Since arcs are homeomorphic to $[0,1]$, we can use the Tietze Extension Theorem to extend g continuously from $N_i \cup (\bigcup_{j, j \neq i} \overline{V}_i \cap \overline{V}_j)$ to $\overline{V}_i$ so that $g(\overline{V}_i) = A_i$. Doing this for all $1 \leqslant i \leqslant n$, we have a 2ε-map g from $\bigcup_{i=1}^n \overline{V}_i$ onto P so that $g(N) = P$. □

Lemma 4.6. *Suppose that M is a nondegenerate indecomposable continuum, P is a graph, M is P-like, $\varepsilon > 0$ and $n > 0$ is an integer. Then, there are pairwise disjoint, nowhere dense subcontinua $K_1, K_2, \ldots, K_n$ of M and an ε-map g from M onto P such that $g(K_i) = P$ for all $1 \leqslant i \leqslant n$.*

Proof. This essentially follows from the proof of Lemma 4.5 and that M is an indecomposable continuum. Let f be an ε-map from M onto P. Let V_i's, and η be as in the proof of Lemma 4.5. Using the fact that composants of a continuum are dense in the continuum and a nondegenerate indecomposable continuum has uncountably many pairwise disjoint composants, we may chose pairwise disjoint continua $K_1, K_2, \ldots, K_n$ so that the Hausdorff distance from any K_i to M is less than η. Now we define g as in the earlier proof considering $K_1, K_2, \ldots, K_n$. Then, g restricted to M is the desired function. □

Theorem 4.7 (Krasinkiewicz [8]). *The following are equivalent for a continuum M:*

- *M is hereditarily indecomposable.*
- *If C, D are two disjoint closed subsets of M and U is an open set intersecting each component of C, then there are closed sets H, K such that $C \subseteq H$, $D \subseteq K$, $M = H \cup K$, and $H \cap K \subseteq U \setminus (C \cup D)$.*

Theorem 4.8. *Let P be a graph and M be a nondegenerate hereditarily indecomposable continuum which is P-like. Suppose that Q is such that $P \prec_3 Q$. Then, M is Q-like.*

Proof. Let b_1, b_2 be two distinct endpoints of P and A be an arc with endpoints b_1, b_2 such that $A \cap P = \{b_1, b_2\}$. Let $A_1 = [a, b_1]$, $A_2 = [a, b_2]$ be two arcs of P such that $A_1 \cap A_2 = \{a\}$ and $(a, b_1]$ and $(a, b_2]$ are open in P. Let $Q = P \cup A$. Let $\varepsilon > 0$. Let $\delta > 0$ and f be a continuous map from M onto P so that if U is a set with diameter less than δ, then $f^{-1}(U)$ has diameter less than $\frac{\varepsilon}{2}$. As M is a nondegenerate hereditarily indecomposable continuum, by Lemma 4.6 we may assume that there are three pairwise disjoint continua K_1, K_2, K_3 in M such that $f(K_i) = P$, $1 \leqslant i \leqslant 3$. For $i = 1, 2$, let $T_i = [a_i, b_i] \subseteq A_i$ be an arc such that $a_i \neq a$ and the diameter of $[a, a_1] \cup [a, a_2]$ is less than δ.

Let $b \in A \setminus \{b_1, b_2\}$. Let h_i be a homeomorphism from T_i onto the subarc of A determined by b_i and b such that $h_i(a_i) = b$ and $h_i(b_i) = b_i$. Let $h = h_1 \cup h_2$. Extend h to $A_1 \cup A_2$ by letting $h(x) = b$ for $x \in [a, a_1] \cup [a, a_2]$. Then, $h : (A_1 \cup A_2) \to A$ is a continuous map.

Let $R = P \setminus ((a, b_1] \cup (a, b_2])$. Let $D = K_3 \cup f^{-1}(R)$. Then, D is a compact subset of M since R is closed. Note that for each $i = 1, 2$, $O_i = f^{-1}((a_i, b_i]) \cap K_i$ is a relatively open subset of K_i. Let $p_i \in O_i$ be such that $f(p_i) = b_i$. Now by the Boundary Bumping Theorem we may choose a continuum $N_i \subseteq \overline{O}_i$ such that $p_i \in N_i$ and $N_i \cap \partial(O_i) \neq \emptyset$. Then, $f(N_i) = [a_i, b_i] = T_i$. We let $C = N_1 \cup N_2$. We observe that $C \cap D = \emptyset$.

For $i = 1, 2$, let $J_i = [a_i', b_i] \subseteq T_i$ be such that $\operatorname{diam}(J_i) < \delta$ and let $U_i = f^{-1}((a_i', b_i])$. Then, $\operatorname{diam}(\overline{U}_i) < \frac{\varepsilon}{2}$ and $U_1 \cap U_2 = \emptyset$. Let $U = U_1 \cup U_2$. Note that U intersects each component of C since $p_i \in U_i \cap N_i$ for $i = 1, 2$.

We now apply the Krasinkiewicz Theorem to M, C, D, and U. Let H, K be closed sets such that $M = H \cup K$, $C \subseteq H$, $D \subseteq K$ and $H \cap K \subseteq U$. Note that $H \setminus U$ and $K \setminus U$ are disjoint closed sets whose union is $M \setminus U$. Let us proceed to define an ε-map g from M onto Q. We do this by considering three cases.

The first case is that $x \in (H \setminus U)$. In this case, $f(x) \in A_1 \cup A_2$, since $f^{-1}(R) \subseteq D \subseteq K$ and $H \cap K \subseteq U$. Define $g(x) = h(f(x))$. Note that $g(H \setminus U) \subseteq A$. The second case is $x \in (K \setminus U)$. In this case, let $g(x) = f(x)$. Finally, consider the case $x \in U$. We define g on U_1 first. Note that g is well defined on $\partial(U_1)$ and $g|_{M \setminus U}$ is continuous. Since $f(\overline{U}_1) \subseteq J_1$, we have that $g(\partial(U_1)) \subseteq J_1 \cup h(J_1)$. Note that $J_1 \cup h(J_1)$ is an arc. By the Boundary Bumping Theorem, we know that U_1 contains a nondegenerate continuum. Hence, by the Tietze Extension Theorem, we can extend g continuously from the Boundary of U_1 to all of $\overline{U}_1$ so that $g(\overline{U}_1) = J_1 \cup h(J_1)$. We define g analogously on U_2. Now we have a continuous function g from M into Q.

Let us first show that g maps onto Q. Note that

$$P \setminus (J_1 \cup J_2) \subseteq f(K_3 \setminus U) = g(K_3 \setminus U).$$

Also,

$$\bigcup_{i=1}^{2} \bigl(J_i \cup h(J_i)\bigr) \subseteq g\bigl(\overline{U}\bigr).$$

Finally,

$$A\Bigg\backslash\left(\bigcup_{i=1}^{2}h(J_i)\right)\subseteq\bigcup_{i=1}^{2}h\big(f(N_i\setminus U_i)\big)\subseteq g(C\setminus U).$$

Hence it follows that g maps onto Q. Now we want to verify that g is an ε-map. Let $y\in Q$. We first note that

$$\begin{aligned}g^{-1}(y)&=(g|_{H\setminus U})^{-1}(y)\cup(g|_{K\setminus U})^{-1}(y)\cup(g|_U)^{-1}(y)\\&=[h\circ f|_{H\setminus U}]^{-1}(y)\cup(f|_{K\setminus U})^{-1}(y)\cup(g|_U)^{-1}(y).\end{aligned}$$

If $y\in P\setminus(J_1\cup J_2)$, then $g^{-1}(y)=(f|_{K\setminus U})^{-1}(y)\subseteq f^{-1}(y)$ and hence has diameter less than ε. Now suppose that $y\in(J_i\cup h(J_i))$ for $i=1$ or $i=2$. Then, $g^{-1}(y)\subseteq f^{-1}(J_i)\cup U_i$. Since $p_i\in f^{-1}(J_i)\cap U_i$ and each of $f^{-1}(J_i)$ and U_i has diameter less than $\frac{\varepsilon}{2}$, we have that $f^{-1}(y)$ has diameter less than ε. Now we consider the case that $y\in A\setminus(h(J_1)\cup h(J_2)))$. If $y\neq b$, then $g^{-1}(y)=(h\circ f|_{H\setminus U})^{-1}(y)\subseteq f^{-1}(h^{-1}(y))$ and hence has diameter less than ε. Finally, consider the case $y=b$. Then, $g^{-1}(y)=f^{-1}([a,a_1]\cup[a,a_2])$. Since the diameter of $[a,a_1]\cup[a,a_2]$ is less than δ, we have that the diameter of $f^{-1}([a,a_1]\cup[a,a_2])=g^{-1}(y)$ is less than ε. □

Theorem 4.9. *Let P be a graph and M be a nondegenerate hereditarily indecomposable continuum which is P-like. Suppose that Q is such that $P\prec_2 Q$. Then, M is Q-like.*

Proof. Assume the hypothesis. Let $[a,b]$ be an arc of P with b an endpoint of P and $(a,b]$ open in P. Let A be an arc with endpoints a,b such that $A\cap P=\{a,b\}$. Let $Q=P\cup A$. Let $c\in A\setminus\{a,b\}$. By Theorem 4.4, M is $P\cup A_1$ like where A_1 is the subarc of A determined by a and c. Now applying Theorem 4.8 to $P\cup A_1$ and its endpoints c,b, we get that M is Q-like. □

Corollary 4.10. *Suppose that M is a nondegenerate hereditarily indecomposable continuum which is P-like for some graph P and Q is a graph which can be obtained from P by applying a finite sequence of extensions of method I, II or III. Then, M is Q-like.*

Proof. This simply follows from applying Theorems 4.4, 4.9, and 4.8. □

Corollary 4.11. *Let P be a nondegenerate subcontinuum of the figure-eight. If M is a hereditarily indecomposable continuum which is P-like, then M is figure-eight like.*

5. The fiber structure of a generic map in $C(S^2,I)$

In this section we give a more precise description of the fibers of a generic map $f\in C(S^2,I)$. This section is divided into four subsections. In Subsection 5.1, we construct a well-behaved class of continuous functions in $C(S^2,I)$. We also show in this section that the saddle points are "stable" in $C(S^2,I)$. These results are used in later subsections to determine the fiber structure of a generic map in $C(S^2,I)$. In Subsection 5.2 we show that a generic $f \in C(S^2, I)$ has the property that each component of each fiber of f is either a point or figure eight like. In Subsection 5.3 we show that a generic $f \in C(S^2, I)$ has the property that there is a countable dense set $D \subseteq f(S^2)$ such that for each $y \in D$, $f^{-1}(y)$ has a component which is a Lakes of Wada continuum. In Subsection 5.4 we show that all components of almost every fiber of a generic map in $C(S^2, I)$ are either points, pseudoarcs or pseudocircles.

5.1. A well-behaved countable dense subset of $C(S^2, I)$

We use the following parametrization of S^2. A point $p \in S$ can be represented as

$$p = \Psi(\phi, \theta) = \big(\cos(\theta)\cos(\phi), \cos(\theta)\sin(\phi), \sin(\theta)\big).$$

We denote by $P_{\text{North}} = (0, 0, 1)$, and $P_{\text{South}} = (0, 0, -1)$, the North and South poles of our sphere.

In the sequel we need to consider some triangulations of S^2. Our triangles on S^2 will be either one-to-one Ψ images of a triangle T in $\mathbb{R}^2$, or when P_{North} (or P_{South}) is one of the vertices, images of the form $\Psi([\phi_1, \phi_2] \times [\theta_0, \pi/2])$ (or $\Psi([\phi_1, \phi_2] \times [-\pi/2, \theta_0])$) such that Ψ is one-to-one on $[\phi_1, \phi_2] \times [\theta_0, \pi/2)$ (or on $[\phi_1, \phi_2] \times (-\pi/2, \theta_0]$).

Let us fix an equilateral triangle S in $\mathbb{R}^2$ of side length one with vertices S_1, S_2, S_3. For each triangle T of our triangulation of S^2 with vertices $V_1, V_2, V_3 \in S^2$ we fix a homeomorphism Φ_T from T onto S so that if $x \in V_iV_j$, then $\Phi_T(x) \in S_iS_j$ and the Euclidean distance between $\Phi_T(x)$ and S_i is $\frac{\text{arc length of } (V_i, x)}{\text{arc length of } (V_i, V_j)}$. We say that a function $f : T \to \mathbb{R}$ is *linear* if $f \circ \Phi_T^{-1}$ is linear on S.

Let $K > 1$ be a large odd integer. Next we want to define a triangulation $\mathcal{T} = \mathcal{T}_K$ of S^2. The North cap and the South cap of S^2, respectively, are

$$B_{\text{North}} = \left\{\Psi(\phi, \theta)\colon\ \phi \in [0, 2\pi),\ \frac{\pi}{2} - \frac{\pi}{2K} \leqslant \theta \leqslant \frac{\pi}{2}\right\}, \quad \text{and}$$
$$B_{\text{South}} = \left\{\Psi(\phi, \theta)\colon\ \phi \in [0, 2\pi),\ -\frac{\pi}{2} \leqslant \theta \leqslant -\frac{\pi}{2} + \frac{\pi}{2K}\right\}.$$

Denote by C^2 the closure of $S^2 \setminus (B_{\text{North}} \cup B_{\text{South}})$. Clearly, C^2 is homeomorphic to a cylinder. First, we divide C^2 into non-overlapping "rectangles". If $i \in \{0, \dots, K-1\}$ and j is even, $-K < j < K-1$, we put

$$R(i, j) = \left\{\Psi(\phi, \theta)\colon\ \frac{2\pi i}{K} \leqslant \phi \leqslant \frac{2\pi(i+1)}{K},\ \frac{\pi j}{2K} \leqslant \theta \leqslant \frac{\pi(j+1)}{2K}\right\}.$$

If $i \in \{0, \dots, K-1\}$ and j is odd, $-K < j < K-1$, we put

$$R(i, j) = \left\{\Psi(\phi, \theta)\colon\ \frac{2\pi(i+\frac{1}{2})}{K} \leqslant \phi \leqslant \frac{2\pi(i+\frac{3}{2})}{K},\ \frac{\pi j}{2K} \leqslant \theta \leqslant \frac{\pi(j+1)}{2K}\right\}.$$

The center (the intersection of its "diagonals") of $R(i, j)$ will be denoted by $C(i, j)$. If j is even we call the rectangle even, if j is odd we call the rectangle odd.

First, we triangulate C^2. Given a rectangle $R = R(i, j)$ with vertices V_k, $k = 1, \dots, 4$, we assume that V_1, and V_2 (and hence V_3 and V_4) have the same θ coordinate, see Fig. 1.

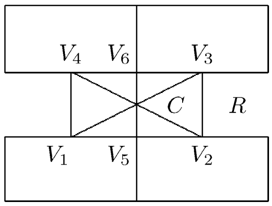

Fig. 1. Rectangle R and its neighbors.

We can also assume that the θ coordinate of V_1 is less than that of V_3. Denote by V_5 and V_6 the midpoint of the side V_1V_2 and V_3V_4, respectively. Taking a component of $\Psi^{-1}(R(i,j))$ consider the triangles determined by $\Psi^{-1}(V_j)$ ($j = 1, \ldots, 6$) and $\Psi^{-1}(C)$, where C is the center of $R(i,j)$. Use the Ψ image of these triangles for the triangulation of $R(i,j)$. We do this on all rectangles $R(i,j)$.

Next we triangulate B_{North} by using the non-overlapping triangles determined by P_{North} and the points $\{\Phi(\frac{\pi i}{K}, \frac{\pi}{2} - \frac{\pi}{2K})\colon i = 0, \ldots, 2K-1\}$. We triangulate B_{South} similarly.

Now we have a *triangulation* of the whole sphere S^2. Denote the vertices of the graph determined by this triangulation by $\mathcal{V}$ and the edges of this graph by $\mathcal{E}$. We use $(\mathcal{V}, \mathcal{E})$ to denote this graph. Observe that apart from the North and South Pole the above graph has degree no more than six at each vertex in $\mathcal{V}$. We denote by $\mathcal{T}$ the set of all closed spherical triangles determined by our triangulation. A triangulation is ε-*fine* if $\operatorname{diam}(T) < \varepsilon$ for all $T \in \mathcal{T}$.

Lemma 5.1. *The set of vertices $\mathcal{V}$ can be divided into three sets: $\mathcal{V}_p$ is the set consisting of the North and the South pole, $\mathcal{V}_4$ is the set of points $V \in \mathcal{V}$ where $(\mathcal{V}, \mathcal{E})$ has degree four at V, and $\mathcal{V}_6$ is the set of points $V \in \mathcal{V}$ where $(\mathcal{V}, \mathcal{E})$ has degree six at V.*

Proof. It follows from the construction that if V is a vertex which is not adjacent to one of the poles, then $(\mathcal{V}, \mathcal{E})$ has degree six at V. If V is adjacent to one of the poles and V is a corner of some rectangle, then $(\mathcal{V}, \mathcal{E})$ has degree six at V as well. If V is adjacent to one of the poles and V is not a corner of some rectangle, then $(\mathcal{V}, \mathcal{E})$ has degree four at V. $\square$

A function $f : S^2 \to \mathbb{R}$ is a *triangular* function if there is a triangulation $\mathcal{T}$ of S^2 so that f is linear on each $T \in \mathcal{T}$, and

$$V, V' \in \mathcal{V},\ V \neq V' \implies f(V) \neq f(V'). \tag{1}$$

We use **TF** to denote the class of triangular functions. The following property holds for all triangular functions f.

Lemma 5.2. *Suppose that $f \in$ **TF**, $\mathcal{T}$ is the triangulation associated to f, and $f^{-1}(y) \cap T \neq \emptyset$ for some $T \in \mathcal{T}$. Then, one of the following is true:*

- *$f^{-1}(y) \cap T$ has exactly one point and this point is a vertex of T;*
- *$f^{-1}(y) \cap T$ is an arc, contains exactly one vertex, and the endpoints of $f^{-1}(y) \cap T$ are the vertex and a point on the side of T opposite to this vertex;*
- *$f^{-1}(y) \cap T$ is an arc intersecting only two sides of T and containing no vertex of T.*

Proof. To see this, consider $f \circ \Phi_T^{-1}$. Function $f \circ \Phi_T^{-1}$ is linear on S, and different vertices of S map to different numbers. Hence, the lemma is satisfied for $f \circ \Phi_T^{-1}$ with respect to S. The validity of the lemma now follows from this fact and that Φ_T is an appropriate homeomorphism from T onto S. $\square$

We say that $f \in \mathbf{TF}$ is *proper* if the following properties hold:

- If $T \in \mathcal{T}$ and $V \in T$ is either the North pole or the center of an even rectangle, then f restricted to T has a maximum at V.
- If $T \in \mathcal{T}$ and $V \in T$ is either the South pole or the center of an odd rectangle, then f restricted to T has a minimum at V.

Lemma 5.3. *Suppose that f is a proper triangular function and $\mathcal{T}$ is the triangulation associated to f. If V is a vertex of $\mathcal{T}$ which is not one of the poles nor the center of some rectangle, then there are vertices $V_i \in \mathcal{V}$, $i = 1, 2$, distinct from V, adjacent to V and $T_i \in \mathcal{T}$, $i = 1, 2$ with $\{V, V_i\} \subseteq T_i$, such that $f|T_1$ has maximum at V_1 and $f|T_2$ has minimum at V_2.*

Proof. If V is not adjacent to one of the poles, then this is clear from the definition of f being proper. If V is adjacent to one of the poles, then the lemma follows from the fact that K is an odd integer. $\square$

Lemma 5.4. *Suppose that f is a proper triangular function. Then, the following properties are true*:

(1) *For all $y \in f(S^2)$, $f^{-1}(y)$ has finitely many components.*
(2) *For each $y \in f(S^2)$, every component of $f^{-1}(y)$ is homeomorphic to a circle, with possibly one exception*; *the exception is either a point or is homeomorphic to the figure eight.*
(3) *There are only finitely many points y in the range of f such that $f^{-1}(y)$ has a component which is a point, or homeomorphic to the figure eight.*
(6) *f has a local extremum at $(x, f(x))$ iff x is an isolated point of $f^{-1}(f(x))$.*

Proof. To see the first conclusion, let $y \in f(S^2)$ and $f^{-1}(y) \cap T \neq \emptyset$ for some $T \in \mathcal{T}$. Then, by Lemma 5.2, we have that $f^{-1}(y) \cap T$ is either a point or an arc. Since $\mathcal{T}$ is finite, we have that $f^{-1}(y)$ has finitely many components.

We now proceed to the second conclusion of the lemma. Let $y \in f(S^2)$ and M be a component of $f^{-1}(y)$. We consider three cases. The first case is that M contains no point of $\mathcal{V}$. Let $M \cap T \neq \emptyset$ for some $T \in \mathcal{T}$. Then, by Lemma 5.2 we have that $M \cap T$ is an arc which intersects two edges of T. Hence, it follows that M is a graph which has degree two. By [13, Corollary 9.6], it follows that M is homeomorphic to a circle.

If M contains the center of some rectangle, the North pole or the South pole, then f has a strict local extremum at this point and hence M is a singleton set.

Finally, let us assume that M contains a vertex V which is not one of the poles nor the center of some rectangle. Since different vertices map to different values, M contains

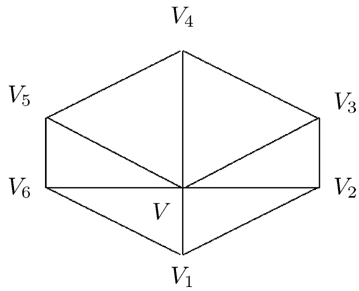

Fig. 2. The triangles meeting at V.

exactly one vertex. Therefore, if $x \in M \setminus \{V\}$, then M has degree two at x. Let us now determine the degree of M at V. By Lemma 5.1, it follows that $(\mathcal{V}, \mathcal{E})$ has degree six or four at V. We will argue that if $(\mathcal{V}, \mathcal{E})$ has degree six at V, then M has degree four or two at V. An analogous argument will show that if $(\mathcal{V}, \mathcal{E})$ has degree four at V, then the degree of M at V is two. Hence, we will obtain that M is homeomorphic either to the figure-eight, or to the circle.

Denote the edges meeting at V by E_j $(j = 1, \dots, 6)$, and denote by V_j the endpoint of E_j which is different from V, see Fig. 2. It is not difficult to see that among the V_j's three are rectangle centers (P_{North} and P_{South} are regarded to be "rectangle centers" in this context). Without limiting generality we can assume that E_j $(j = 1, \dots, 6)$ are labelled in a counterclockwise order and V_1, V_3, and V_5 are the rectangle centers. Then, these are strict local extrema of f and by our choice of even–odd rectangles there are both local minima and maxima among them. Without limiting generality we assume V_1 is a strict local maximum, V_3 and V_5 are local minima. Note that $f(V_1) > y$, $f(V_3) < y$ and $f(V_5) < y$. From the linearity of f on each $T \in \mathcal{T}$ and the fact that f is proper, it follows that $f^{-1}(y) \cap (\bigcup_{i=1}^{6} E_i) = \{V\}$. Since $f(V_3) < y < f(V_1)$, it follows that $f^{-1}(y)$ intersects only one of the edges V_1V_2 or V_2V_3. The same holds for edges V_1V_6 or V_5V_6. Now let us look at V_4. If $f(V_4) > y$, then $f^{-1}(y)$ intersects each of the edges V_3V_4 and V_4V_5. If $f(V_4) < y$, then $f^{-1}(y)$ intersects neither of these edges. By Lemma 5.2 it follows that M has degree four at V if $f(V_4) > y$ and M has degree two at V if $f(V_4) < y$.

To verify the third conclusion of the lemma we notice that $f^{-1}(y)$ has a component which is a point, or homeomorphic to figure-eight if and only if $y \in f(\mathcal{V})$. As $f(\mathcal{V})$ is finite, the third conclusion holds.

Finally, let us verify the fourth conclusion. If $x \in S^2 \setminus \mathcal{V}$, then f does not have a local extremum at x as different vertices of $\mathcal{V}$ map to different numbers under f, and f is linear on some $T \in \mathcal{T}$ which contains x. If $x \in \mathcal{V}$ but x is not the center of some rectangle $R(i, j)$, then by Lemma 5.3 there are vertices V, V' different from x and $T, T' \in \mathcal{T}$ with $\{x, V\} \subseteq T$ and $\{x, V'\} \subseteq T'$ such that $f|T$ has a maximum at V and $f|T'$ has a minimum at V'. Hence, f does not have a local extremum at x. Therefore, f has a local extremum only at a center of some rectangle and, therefore, it is an isolated point of $f^{-1}(f(x))$. $\square$

Definition 5.5. We say that $f \in C(S^2, I)$ is well-behaved, denoted by $f \in \mathbf{WB}$, if f satisfies the conclusion of Lemma 5.4.

Lemma 5.6. *The set of proper triangular functions is dense in $\mathcal{C}(S^2, I)$. Hence,* **WB** *is dense in $\mathcal{C}(S^2, I)$.*

Proof. We know that $\mathcal{C}(S^2, I)$ is a separable metric space. Given Lemma 5.4, it will suffice to show that for each $h \in \mathcal{C}(S^2, I)$ and $\varepsilon > 0$, there is a proper $f \in \mathbf{TF}$ with $\|f - h\| < 2 \cdot \varepsilon$.

Let us proceed to define f. As h is uniformly continuous, we may choose $\delta > 0$ so that $d(p, q) < \delta,\ p, q \in S^2$ implies $|h(p) - h(q)| < \frac{\varepsilon}{8}$. We choose a triangulation $\mathcal{T}$ of S^2 which is $\frac{\delta}{2}$-fine. We define f on the vertices $\mathcal{V}$ so that if $V, V' \in \mathcal{V}$ with $V \neq V'$, then $f(V) \neq f(V')$ and the following inequalities hold. If $V = P_{\text{North}}$ then we choose $f(P_{\text{North}})$ so that

$$h(P_{\text{North}}) + \frac{\varepsilon}{2} < f(P_{\text{North}}) < h(P_{\text{North}}) + \varepsilon. \tag{2}$$

If $V = P_{\text{South}}$ then we choose $f(P_{\text{South}})$ so that

$$h(P_{\text{South}}) - \varepsilon < f(P_{\text{South}}) < h(P_{\text{South}}) - \frac{\varepsilon}{2}. \tag{3}$$

If V is the vertex of a rectangle R then we choose $f(V)$ so that

$$h(V) - \frac{\varepsilon}{8} < f(V) < h(V) + \frac{\varepsilon}{8}. \tag{4}$$

If V is the center of an even rectangle R then we choose $f(V)$ so that

$$h(V) + \frac{\varepsilon}{2} < f(V) < h(V) + \varepsilon. \tag{5}$$

If V is the center of an odd rectangle R then we choose $f(V)$ so that

$$h(V) - \varepsilon < f(V) < h(V) - \frac{\varepsilon}{2}. \tag{6}$$

If $T \in \mathcal{T}$, then we extend f to T so that f is linear on T. From the linearity of f on each T and the definition of Φ_T, it follows that f is a well-defined continuous function on all of S^2. By our choice of f on $\mathcal{V}$ and by the linearity of f, it follows that f is a proper triangular function and $\|f - h\| < 2 \cdot \varepsilon$. □

We next discuss the stability of the figure-eight in level sets of functions in class **WB**. However, we first need some definitions, notation and background terminology.

Definition 5.7. Suppose that $M \subseteq S^2$ is a compact set and $n \geqslant 2$. We say that M *separates S^2 into at least n pieces* if $S^2 \setminus M$ has at least n components. If $A_1, A_2, \dots, A_n$ are pairwise disjoint subsets of S^2, then we say that M separates $A_1, A_2, \dots, A_n$ means that $A_1, A_2, \dots, A_n$ are subsets of different components of $S^2 \setminus M$.

Remark 5.8. We remark here that for a closed set $M \subseteq S^2$, the components of $S^2 \setminus M$ are open sets which are arcwise connected. Therefore, M separates $A_1, A_2, \dots, A_n$ iff $\bigcup_{i=1}^n A_i \subseteq S^2 \setminus M$ and there is no arc in $S^2 \setminus M$ which intersects two of the A_i's.

Lemma 5.9. *Suppose that $U_1, U_2, \dots, U_n$ are pairwise disjoint nonempty open subsets of S^2 and $\{M_k\}_{k=1}^{\infty}$ is a sequence of compact sets each of which separates $U_1, U_2, \dots, U_n$.*

Furthermore, assume that $\{M_k\}$ *converges to* M *in the Hausdorff metric. Then,* M *separates* $U_1, U_2, \dots, U_n$ *and if* M *is a subset of the compact set* $N \subseteq S^2$ *then* N *separates* S^2 *into at least* n *pieces or* N *contains at least one of* $U_1, U_2, \dots, U_n$.

Proof. To show that M separates $U_1, U_2, \dots, U_n$, we use Remark 5.8. We first note that $M \cap (\bigcup_{i=1}^n U_i) = \emptyset$ as $\{M_k\}$ converges to M in the Hausdorff metric and U_i's are open sets. Let $1 \leqslant i < j \leqslant n$. Let A be an arc in S^2 which intersects U_i and U_j. Since M_k separates U_i from U_j, we have that $M_k \cap A \neq \emptyset$ for all k. Since $\{M_k\}$ is a sequence of compact sets which converges to M in the Hausdorff metric and A is compact as well, we have that $M \cap A \neq \emptyset$. Hence, M separates $U_1, U_2, \dots, U_n$. Now let N be a compact set such that $M \subseteq N$. Let $V_i = U_i \setminus N$ for $1 \leqslant i \leqslant n$. If $V_i = \emptyset$ for some i, then we have that $U_i \subseteq N$ for some i. Hence, assume that no V_i is empty. Then, $\bigcup_{i=1}^n V_i \subseteq S^2 \setminus N$ and every arc in S^2 intersecting V_i and V_j with $i \neq j$ must intersect N because each such arc must intersect M. Hence, we have that N separates $V_1, V_2, \dots, V_n$ and $S^2 \setminus N$ has at least n components. □

Remark 5.10. Let $f \in \mathbf{WB}$. Consider the function $F : f(S^2) \to \mathcal{K}(S^2)$ defined by $F(y) = f^{-1}(y)$. Then, F is continuous at all y which are not local extrema of f. Hence, F is continuous except at a finite set. At each such exception y, we have that $\lim_{t\to y^+} F(t)$ and $\lim_{t\to y^-} F(t)$ exist the symmetric difference of these two sets is a singleton set where f attains a local extrema.

Definition 5.11. Suppose that $f \in \mathbf{WB}$, $y \in f(S^2)$, and $A_i \in S^2 \setminus f^{-1}(y)$, $1 \leqslant i \leqslant 3$ are disjoint continua. Let $\mathcal{M}_{A_2}$ be the set of those components M of $f^{-1}(y)$ which separate A_2 from A_3 and A_2 from A_1. Similarly, let $\mathcal{M}_{A_3}$ be the set of those components M of $f^{-1}(y)$ which separate A_3 from A_2 and separate A_3 from A_1. Furthermore, assume that $\mathcal{M}_{A_2} \neq \emptyset$ and $\mathcal{M}_{A_3} \neq \emptyset$. Then, we let

$$\delta(f, y, A_1, A_2, A_3) = \inf\bigl\{d_H(M_2, M_3)\colon\ M_2 \in \mathcal{M}_{A_2} \text{ and } M_3 \in \mathcal{M}_{A_3}\bigr\}.$$

Remark 5.12. As $f^{-1}(y)$ has finitely many components, we have that the inf in the definition of $\delta(f, y, A_1, A_2, A_3)$ is actually realized. Hence, if $\delta(f, y, A_1, A_2, A_3) = 0$, then we have that there is a component of $f^{-1}(y)$ which separates A_1, A_2, A_3. As $f \in \mathbf{WB}$, it follows that $f^{-1}(y)$ contains a component homeomorphic to the figure-eight.

Remark 5.13. If $\delta(f, y, A_1, A_2, A_3) > 0$, then we have $\delta(f, y', A_1, A_2, A_3) > 0$ for all y' sufficiently close to y. To see this, let us first recall the fact that the set of all t such that $f^{-1}(t)$ contains a component not homeomorphic to the circle is finite. Therefore, for t sufficiently close to y with $t \neq y$, we have that each component of $f^{-1}(t)$ is homeomorphic to the circle. Therefore, it separates S^2 into exactly two pieces. Hence, for y' sufficiently close to y, we have that $\delta(f, y', A_1, A_2, A_3)$ is well-defined by Remark 5.10. By Remark 5.12, we have that $\delta(f, y', A_1, A_2, A_3) \neq 0$.

Lemma 5.14. *Suppose that* $f \in \mathbf{WB}$ *and* $f^{-1}(y)$ *contains a component* M *homeomorphic to the figure-eight for some* $y \in \mathbb{R}$ *and* $\gamma > 0$. *Then, there is* $0 < \varepsilon < \gamma$ *such that if* $g \in$

$\mathcal{C}(S^2, I)$ and $\|f - g\| < \varepsilon$, then there is $y' \in \mathbb{R}$ with $|y - y'| \leqslant 2\varepsilon$ such that $g^{-1}(y')$ contains a component which separates S^2 into at least three pieces.

Proof. Let M be the component of $f^{-1}(y)$ which is homeomorphic to the figure-eight. Then, $S^2 \setminus M$ has three components which we denote by W_i, $1 \leqslant i \leqslant 3$. As $f \in \mathbf{WB}$, we have that $f^{-1}(y)$ has finitely many components and M contains no point of local extremum of f. Hence, we can choose an open set U such that

- $M \subseteq U$,
- $\overline{U} \cap (f^{-1}(y) \setminus M) = \emptyset$, and
- for each $1 \leqslant i \leqslant 3$, we have that $f(W_i \cap U) \subseteq (y, \infty)$ or $f(W_i \cap U) \subseteq (-\infty, y)$.

(Imagine U as a small "fattening up" of M.) Since f does not have a local extremum at any point of M, we have that for some i, $f(W_i \cap U) \subseteq (y, \infty)$ and for another i, $f(W_i \cap U) \subseteq (-\infty, y)$. Without loss of generality, we assume that $f(W_1 \cap U) \subseteq (y, \infty)$ and $f(W_2 \cap U) \cup f(W_3 \cap U) \subseteq (-\infty, y)$. Using Tietze's Extension Theorem, we define a function f_0 such that $f_0 = f$ on U and $f_0(W_1) \subseteq (y, \infty)$ and $f_0(W_2 \cup W_3) \subseteq (-\infty, y)$. Let $\rho_0 = \inf\{|f(x) - y|\colon x \in \partial U\} > 0$. Let $U_i \subseteq U$ be an open ball (hence $\overline{U}_i$ is a continuum) such that $\overline{U}_i \subseteq W_i$. Set $\rho_1 = \inf\{|f_0(x) - y|\colon x \in \bigcup_{i=1}^3 W_i \setminus U\}$ and $\rho = \min\{\rho_0, \rho_1\}$. Note that $\rho_1 > 0$ and hence $\rho > 0$. Let

$$\varepsilon = \frac{1}{8} \min\bigl\{\gamma, \rho, \inf f(U_1) - y, y - \sup f(U_2), y - \sup f(U_3)\bigr\}.$$

Then, $\varepsilon > 0$.

Let us first prove the lemma for f_0 and $g_0 \in \mathbf{WB}$. Let $g_0 \in \mathbf{WB}$ such that $\|f_0 - g_0\| < \varepsilon$. Let $\varepsilon < \varepsilon' < 2\varepsilon$ be such that $g_0^{-1}(y - \varepsilon')$ contains no component homeomorphic to the figure-eight or to a point. By our choice of ε, we have that for $y - 7\varepsilon \leqslant z \leqslant y + 7\varepsilon$, $g_0^{-1}(z) \cap \bigcup_{i=1}^3 \overline{U}_i = \emptyset$. As $g_0(M) \subseteq (y - \varepsilon, y + \varepsilon)$, each arc intersecting $\overline{U}_2$ and $\overline{U}_3$ intersects M and each arc intersecting $\overline{U}_2$ and $\overline{U}_1$ intersects M, we have that $g^{-1}(y - \varepsilon')$ separates $\overline{U}_2$ from $\overline{U}_3$ and $g^{-1}(y - \varepsilon')$ separates $\overline{U}_2$ from $\overline{U}_1$. Let C_2 be the boundary of the component of $S^2 \setminus g^{-1}(y - \varepsilon')$ containing $\overline{U}_2$. Note that C_2 is homeomorphic to a circle. Hence, C_2 separates $\overline{U}_2$ from $\overline{U}_3$ and separates $\overline{U}_2$ from $\overline{U}_1$. Similarly, there is a component C_3 of $g_0^{-1}(y - \varepsilon')$ which separates $\overline{U}_3$ from $\overline{U}_2$ and separates $\overline{U}_3$ from $\overline{U}_1$. Hence, $\delta(g_0, y - \varepsilon', \overline{U}_1, \overline{U}_2, \overline{U}_3)$ is well-defined and by Remark 5.12, $\delta(g_0, y - \varepsilon', \overline{U}_1, \overline{U}_2, \overline{U}_3) > 0$.

Let us next observe that for $z \geqslant y + \varepsilon$, $g_0^{-1}(z)$ does not separate $\overline{U}_2$ from $\overline{U}_3$. Indeed, this is the case as $g_0(W_2) \cup g_0(W_3) \cup g_0(M) \subseteq (-\infty, y + \varepsilon)$ and $W_2 \cup W_3 \cup M$ is connected.

Let

$$T = \bigl\{t\colon \delta\bigl(g_0, t, \overline{U}_1, \overline{U}_2, \overline{U}_3\bigr) > 0\bigr\}.$$

Then, $y - \varepsilon' \in T$ and $y' = \sup T$ is less than $y + \varepsilon$. Hence $|y - y'| < 2\varepsilon$. By Remark 5.10, $\delta(g_0, y', \overline{U}_1, \overline{U}_2, \overline{U}_3)$ is well-defined and by Remark 5.13, we have that $\delta(g_0, y', \overline{U}_1, \overline{U}_2, \overline{U}_3) = 0$. Hence, by Remark 5.12 $g_0^{-1}(y')$ has a component which is homeomorphic to the figure-eight which separates $\overline{U}_1, \overline{U}_2, \overline{U}_3$. By the choice of ρ and ε we also have $g_0^{-1}(y') \subseteq U$.

Let us now consider $g_0 \in \mathcal{C}(S^2, I)$ with $\|f_0 - g_0\| < \varepsilon$. As **WB** is dense in $\mathcal{C}(S^2, I)$ (Lemma 5.6), we may choose a sequence $\{g_n\}$ $(n = 1, 2, \ldots)$ in **WB** with $\|f_0 - g_n\| < \varepsilon$ and g_0 is the uniform limit of $\{g_n\}$. By what we have just shown, there is y_n' with $|y_n' - y| < 2\varepsilon$ and a component M_n of $g_0^{-1}(y_n')$ which separates U_1, U_2, U_3. By turning to a suitable subsequence, we may assume that $\{M_n\}$ converges to some continuum M' in the Hausdorff metric and $\{y_n'\}$ converges to some y'. Note that $g_0(M') = y'$ and $y' \in [y - 2\varepsilon, y + 2\varepsilon]$. By Lemma 5.9, N, the component of $g_0^{-1}(y')$ containing M' either separates U_1, U_2, U_3 or it contains one of the U_i's. As $g_0(U_i) \cap [y - 2\varepsilon, y + 2\varepsilon] = \emptyset$ for $1 \leqslant i \leqslant 3$, we have that N separates U_1, U_2, U_3. Again $N \subseteq U$ by the choice of ρ and ε.

Finally, let us consider the general case of $f \in$ **WB** and $g \in \mathcal{C}(S^2, I)$ with $\|f - g\| < \varepsilon$. Let U and f_0 be as above. Then, using Tietze's Extension Theorem (applied to $G = (g - f)|_{\overline{U}}$) one can choose G_0 such that G_0 equals $g - f$ on U and $\|G_0\| < \varepsilon$. Then set $g_0 = f_0 + G_0$. Now $g_0 = g$ on U and $\|g_0 - f_0\| < \varepsilon$. By the above argument we can find a component, $N \subseteq U$ of $g_0^{-1}(y')$ which separates U_1, U_2 and U_3. Since $g_0 = g$ on U we obtain that N is a component of $g^{-1}(y')$ as well. $\square$

5.2. *Figure-eight-likeness in generic maps*

In this subsection we show that a generic $f \in C(S^2, I)$ has the property that all components of each fiber of f are either points, or figure-eight-like.

Let us first note that up to homeomorphism, there are only finitely many subcontinua of figure-eight, namely a point, an arc, a circle, the letter T, the letter X, a circle with one hair, a circle with two hairs originating from the same point, and the figure-eight. We let $\mathcal{T}$ be this finite collection.

Let us also recall the following result which was proved independently by Krasinkiewicz [9] and Levin [10].

Theorem 5.15 (Krasinkiewicz–Levin). *Let X be a compact metric space. Then, a generic $f \in C(X, I)$ has the property that each of its fibers is a Bing compactum, a compactum with all components hereditarily indecomposable.*

Lemma 5.16. *A generic $f \in \mathcal{C}(S^2, I)$ has the property that if $y \in f(S^2)$ and M is a non-degenerate component of $f^{-1}(y)$, then M is hereditarily indecomposable and P-like for some $P \in \mathcal{T}$.*

Proof. That M is hereditarily indecomposable follows from Theorem 5.15. Let us now show that M is P-like for some $P \in \mathcal{T}$. Let $\varepsilon > 0$. Let $\mathcal{G}$ be the collection of those $f \in \mathcal{C}(S^2, I)$ for which there is $y \in f(S^2)$ and a component M of $f^{-1}(y)$ such that there is no ε-map from M onto any element of $\mathcal{T}$. It will suffice to show that the closure of $\mathcal{G}$ is nowhere dense. Let $\{f_n\}$ be a sequence of elements in $\mathcal{G}$ and let f be its limit in the sup norm. Let $\{y_n\}$ and $\{M_n\}$ be such that M_n is a component of $f_n^{-1}(y_n)$ and there is no ε-map from M_n onto any member of $\mathcal{T}$. Without loss of generality we may assume that $\{y_n\}$ converges to some y and $\{M_n\}$ converges to some M in the Hausdorff metric. Then, $f(M) = y$. Let N be the component of $f^{-1}(y)$ which contains M. We claim that there is no $\frac{\varepsilon}{2}$-map from N onto any member of $\mathcal{T}$. To obtain a contradiction, assume there is an

$\frac{\varepsilon}{2}$-map from N onto some member of $\mathcal{T}$. As M is a subcontinuum of N, there is an $\frac{\varepsilon}{2}$-map from M onto some member of $\mathcal{T}$. By Lemma 4.5, for sufficiently large n, there is an ε-map from M_n onto some element of $\mathcal{T}$, yielding a contradiction. Hence, we have shown that if $g \in \overline{\mathcal{G}}$, then there is $y \in g(S^2)$ and a component M of $g^{-1}(y)$ which is P-like for no $P \in \mathcal{T}$. By Lemma 5.6, we have that $\overline{\mathcal{G}}$ is nowhere dense. $\square$

Theorem 5.17. *A generic $f \in C(S^2, I)$ has the property that each component of each fiber of f is either a point, or a hereditarily indecomposable continuum which is figure-eight-like.*

Proof. This simply follows form Lemma 5.16 and Theorem 4.11. $\square$

5.3. Existence of Lakes of Wada continua

In this subsection we show that a generic $f \in C(S^2, I)$ has the property that there is a countable dense set $D \subseteq f(S^2)$ such that for all $y \in D$, there is a component of $f^{-1}(y)$ which is a Lakes of Wada continuum.

Lemma 5.18. *A generic $f \in \mathcal{C}(S^2, I)$ has the property that if y is in the range of f, then no component of $f^{-1}(y)$ separates S^2 into more than three pieces.*

Proof. Fix a countable basis $\mathcal{B}$ for the topology on S^2. Let U_1, U_2, U_3, U_4 be pairwise disjoint elements of $\mathcal{B}$. Let $\mathcal{G}$ consist of those $f \in \mathcal{C}(S^2, I)$ for which there is a y in its range and a component M of $f^{-1}(y)$ which separates U_1, U_2, U_3, U_4. It will suffice to show that the closure of $\mathcal{G}$ in $\mathcal{C}(S^2, I)$ is a nowhere dense subset of $\mathcal{C}(S^2, I)$. Let $\{f_n\}$ be a sequence in $\mathcal{G}$ and let f be its limit in the sup norm. Let $\{y_n\}$ and $\{M_n\}$ be such that for all n we have that M_n is a component of $f_n^{-1}(y_n)$ and M_n separates U_1, U_2, U_3, U_4. Without loss of generality, we may assume that $\{y_n\}$ converges to some number y and $\{M_n\}$ converges to some continuum M in the Hausdorff metric. We note that $f(M) = y$. Let N be the component of $f^{-1}(y)$ containing M. By Lemma 5.9 we have that N separates S^2 into at least four pieces, or N contains an open set. Therefore, we have that if g is in the closure of $\mathcal{G}$, then there is y and a component of $g^{-1}(y)$ which separates S^2 into at least four pieces, or some component of $g^{-1}(y)$ contains an open set. By Lemma 5.6, it follows that the closure of $\mathcal{G}$ is nowhere dense. $\square$

Lemma 5.19. *Suppose that $M_1, M_2 \subseteq S^2$ are disjoint continua, separating S^2 into k_1, k_2 pieces, respectively. Then, $M_1 \cup M_2$ separates S^2 into at least $k_1 + k_2 - 1$ pieces.*

Proof. We note that since M_2 is a continuum, M_2 is a subset of some component of $S^2 \setminus M_1$. Using this observation, choose components $U_1, U_2, \dots, U_{k_1}$ of $S^2 \setminus M_1$ and components $V_1, V_2, \dots, V_{k_2}$ of $S^2 \setminus M_2$ so that $M_2 \subseteq U_1$. We first note that for all $1 \leqslant i \leqslant k_2$, $\partial V_i \subseteq M_2 \subseteq U_1$. Hence, $V_i \cap U_1 \neq \emptyset$ for $i = 1, 2, \dots, k_2$. Now we claim that $M_1 \cup M_2$ separates $U_2, \dots, U_{k_1}, U_1 \cap V_1, U_1 \cap V_2, \dots, U_1 \cap V_{k_2}$. Clearly, $U_2 \cup \dots \cup U_{k_1} \cup (U_1 \cap V_1) \cup (U_1 \cap V_2) \cup \dots \cup (U_1 \cap V_{k_2}) \subseteq S^2 \setminus (M_1 \cup M_2)$. Let W_1, W_2 be two distinct elements of $\{U_2, \dots, U_{k_1}, U_1 \cap V_1, U_1 \cap V_2, \dots, U_1 \cap V_{k_2}\}$ and let A be an arc in S^2 intersecting W_1 and W_2. If $\{W_1, W_2\} \subseteq \{U_2, \dots, U_{k_1}\}$, then $A \cap M_1 \neq \emptyset$. If $\{W_1, W_2\} \subseteq \{U_1 \cap V_1, U_1 \cap V_2, \dots, U_1 \cap V_{k_2}\}$, then $A \cap M_2 \neq \emptyset$. If one of $\{W_1, W_2\}$ is in $\{U_2, \dots, U_{k_1}\}$ and the other is in $\{U_1 \cap V_1, U_1 \cap V_2, \dots, U_1 \cap V_{k_2}\}$, then we have that $A \cap M_1 \neq \emptyset$. Hence, we have shown that $A \cap (M_1 \cup M_2) \neq \emptyset$. Therefore, $S^2 \setminus (M_1 \cup M_2)$ has at least $k_1 + k_2 - 1$ components. □

Lemma 5.20. *A generic $f \in \mathcal{C}(S^2, I)$ has the property that if y is in the range of f, then $f^{-1}(y)$ contains at most one component which separates S^2 into three pieces or more.*

Proof. As earlier, fix a countable basis $\mathcal{B}$ for the topology on S^2. For $i = 1, 2$, let U_1^i, U_2^i, U_3^i be elements of $\mathcal{B}$ so that $U_k^i \cap U_l^i = \emptyset$ if $k \neq l$. Let $\mathcal{G}$ be the set of those functions $f \in \mathcal{C}(S^2, I)$ for which there is y and two distinct components M^1 and M^2 of $f^{-1}(y)$ such that M^i separates $\{U_1^i, U_2^i, U_3^i\}$. It will suffice to show that the closure of $\mathcal{G}$ is nowhere dense. As before, let $\{f_n\}$ be a sequence in $\mathcal{G}$ and let f be its limit in the sup norm. Let $\{y_n\}$ and $\{M_n^i\}$ $(i = 1, 2)$ be such that for all n and i, we have that M_n^i is a component of $f_n^{-1}(y_n)$, M_n^i separates U_1^i, U_2^i, U_3^i; furthermore M_n^1 and M_n^2 are two distinct components of $f_n^{-1}(y_n)$. Without loss of generality, we may assume that $\{y_n\}$ converges to some number y and $\{M_n^i\}$ converges to some continuum M^i in the Hausdorff metric. We note that $f(M^i) = y$. Let N^i be the component of $f^{-1}(y)$ containing M^i. We have that either N^1 and N^2 are disjoint or $N^1 = N^2$.

Let us first consider the case $N^1 \cap N^2 = \emptyset$. Then, by Lemma 5.9, we have that N^i either separates S^2 into three pieces or contains an open set. In this case, we have that there is y and a component of $f^{-1}(y)$ which contains an open set, or there are two components of $f^{-1}(y)$ which separate S^2 into at least three pieces.

Let us now consider the case that $N^1 = N^2$. Note that for all n, M_n^1 and M_n^2 are two disjoint continua each one of which separates S^2 into at least three pieces. Therefore, by Lemma 5.19, we have that $M_n^1 \cup M_n^2$ separates S^2 into at least five pieces. Since $N^1 \supseteq M^1 \cup M^2$, we have by Lemma 5.9 that N^1 separates the plane into at least five pieces or N^1 contains an open set. In this case, we have that there is y and a component of $f^{-1}(y)$ which separates S^2 into five pieces or contains an open set.

Combining the two cases above, what we have is that if $g \in \overline{\mathcal{G}}$, then there is y such that one of the following happens:

- $g^{-1}(y)$ has at least two distinct components each of which separates S^2 into three or more pieces,
- $g^{-1}(y)$ has at least one component which separates S^2 into five or more pieces, or
- $g^{-1}(y)$ has a component which contains an open set.

Now it follows from Lemma 5.6 that $\overline{\mathcal{G}}$ is nowhere dense. □

Lemma 5.21. *If $\mathcal{M}$ is a pairwise disjoint collection of continua in S^2 and each element of $\mathcal{M}$ separates S^2 into at least three components but no more than finitely many, then $\mathcal{M}$ is countable.*

Proof. To obtain a contradiction, assume that $\mathcal{M}$ is an uncountable such collection. By taking an appropriate uncountable subcollection, we may assume that there is a positive integer $n \geqslant 3$ such that each element of $\mathcal{M}$ separates S^2 into exactly n components. Furthermore, using the separability of S^2, we may assume that there is a sequence of distinct points $x_1, x_2, \dots, x_n \in S^2$ which are separated by each element of $\mathcal{M}$. Let M_1, M_2 be two distinct continua in $\mathcal{M}$ which separate $x_1, x_2, \dots, x_n$. Let $U_1, U_2, \dots, U_n$ be the components of $S^2 \setminus M_1$ which contain $x_1, x_2, \dots, x_n$, respectively. Since M_2 is a continuum, M_2 is a subset of one of $U_1, U_2, \dots, U_n$. Without loss of generality, we may assume that it is U_1. Then, $U_2 \cup U_3 \cup M_1$ is a connected set which misses M_2. This contradicts that M_2 separates x_2 and x_3. Hence, $\mathcal{M}$ is countable. □

Lemma 5.22. *A generic $f \in \mathcal{C}(S^2, I)$ has the property that there is a dense set $D \subseteq f(S^2)$ such that for each $y \in D$, there is a component of $f^{-1}(y)$ which separates S^2 into three pieces or more.*

Proof. Let $r \in I$ be a rational number and n be an integer. Let $\mathcal{G}_{r,n}$ be those functions f such that if $f(S^2) \cap (r - 1/n, r + 1/n) \neq \emptyset$, then there is a $y \in (r - 3/n, r + 3/n)$ such that $f^{-1}(y)$ contains a component which separates the plane into at least three pieces. We claim that $\mathcal{G}_{r,n}$ contains a dense open set in $\mathcal{C}(S^2, I)$. Indeed, let $f \in \mathbf{WB}$ and $\varepsilon > 0$. If $f(S^2) \cap (r - 1/n, r + 1/n) \neq \emptyset$, then we can find $g \in \mathbf{WB}$ so that $\|f - g\| < \varepsilon$ and some $y \in (r - 1/n, r + 1/n)$ so that $g^{-1}(y)$ contains a component homeomorphic to the figure-eight. By Lemma 5.14, we have that there is an open ball containing g which is a subset of $\mathcal{G}_{r,n}$. Now, $\mathcal{G} = \bigcap_{r \in \mathbb{Q}} \bigcap_{n=1}^{\infty} \mathcal{G}_{r,n}$ is the desired dense G_δ set. □

Theorem 5.23. *A generic $f \in \mathcal{C}(S^2, I)$ has the property that there is a countable dense set $D \subseteq f(S^2)$ such that for each $y \in f(S^2)$ every component of $f^{-1}(y)$ separates S^2 into two pieces or less except when $y \in D$. In the latter case, the same applies to each component with one exception which separates S^2 into exactly three pieces.*

Proof. A generic function satisfies the conclusions of Lemmas 5.18, 5.20, and 5.22. Let f be such a function. By Lemma 5.18, for each $y \in f(S^2)$, each component of $f^{-1}(y)$ separates S^2 into three pieces or less. Let D be the set of $y \in f(S^2)$ for which there is a component of $f^{-1}(y)$ which separates S^2 into exactly three pieces. By Lemma 5.21, D is countable and by Lemma 5.22 it is dense in $f(S^2)$. By Lemma 5.20, for each $y \in D$, there is exactly one component of $f^{-1}(y)$ which separates S^2 into three pieces. □

Definition 5.24. Suppose that $M, N \subseteq S^2$, and $\varepsilon > 0$. We say that M *is ε-approximated by* N if for each $x \in M$ there is $y \in N$ such that $d(x, y) < \varepsilon$. We note that if M is ε-approximated by N, N is ε-approximated by M and furthermore M, N are compacta, then $d_H(M, N) < \varepsilon$.

Definition 5.25. Let $M \subseteq S^2$ be a continuum and $\varepsilon > 0$. We say that M is *ε-approximated from the outside* if M is ε-approximated by each component U of $S^2 \setminus M$.

Proposition 5.26. *Let $f \in \mathcal{C}(S^2, I)$ and $\gamma, \varepsilon > 0$. Then, there is $g \in \mathbf{WB}$ such that $\|f - g\| < \gamma$ and for all $y \in g(S^2)$ and $M \in \mathrm{Comp}(g^{-1}(y))$, M is ε-approximated from the outside.*

Proof. By Lemma 5.6 we may choose $h \in \mathbf{WB}$ such that $\|f - h\| < \frac{\gamma}{2}$. We note that if $y \in h(S^2)$, $M \in \mathrm{Comp}(h^{-1}(y))$ which is not homeomorphic to the figure-eight, then M is ε-approximated from the outside. We now suggest how h can be adjusted slightly so that all components of all fibers of h are ε-approximated from the outside. Let $y_1 < y_2 < \cdots < y_n$ be those reals for which $h^{-1}(y_j)$ contains a component homeomorphic to the figure-eight.

Set $\phi(x, y) = ((x-1)^2 + y^2)((x+1)^2 + y^2)$. Then, $\phi^{-1}(1)$ is homeomorphic to the figure-eight.

Choose $\gamma' \in (0, \gamma/4)$ so that $\gamma' < \min\{|y_j - y_k|/4\colon\ j \neq k\}$, and for any j any two different components M, M' of $f^{-1}(y_j)$ belong to different components of $h^{-1}([y_j - \gamma', y_j + \gamma'])$. Assume now that M is a component of $h^{-1}(y_j)$ homeomorphic to the figure eight. Denote by Ω_M the component of $h^{-1}((y_j - \gamma', y_j + \gamma'))$ which contains M. We can also assume that γ' is chosen so small that Ω_M is homeomorphic to the "fat" figure eight $\Omega_{0.5,1.5} = \{(x, y)\colon\ 0.5 < \phi(x, y) < 1.5\}$ and without limiting generality we can assume that $\partial\Omega_M$ contains one component of $h^{-1}(y_j + \gamma')$ and two components of $h^{-1}(y_j - \gamma')$. The first component is denoted by M_+ and the other two components are denoted by $M_{-,1}$ and $M_{-,2}$. Now $\Omega_M \setminus M$ can be split into three regions $\Omega_{M,+}$, $\Omega_{M,-,1}$, and $\Omega_{M,-,2}$ so that one component of the boundary of these regions is M_+, $M_{-,1}$, and $M_{-,2}$, respectively. Hence $f > y_j$ on $\Omega_{M,+}$ and $f < y_j$ on $\Omega_{M,-,1} \cup \Omega_{M,-,2}$ and all these regions are homeomorphic to an annulus. Now, one can replace M by another continuum $M' \subseteq \Omega_M$ so that M' is homeomorphic to the figure-eight, it is ε-approximated from the outside and $\Omega_M \setminus M'$ can be split into three regions $\Omega_{M',+}$, $\Omega_{M',-,1}$, and $\Omega_{M',-,2}$ so that these regions are homeomorphic to an annulus, one component of their boundary is part of M', and the other component of their boundary is M_+, $M_{-,1}$, and $M_{-,2}$, respectively. Choose homeomorphisms $\Psi_+ : \overline{\Omega}_{M',+} \to \overline{\Omega}_{M,+}$, $\Psi_{-,1} : \overline{\Omega}_{M',-,1} \to \overline{\Omega}_{M,-,1}$, $\Psi_{-,2} : \overline{\Omega}_{M',-,2} \to \overline{\Omega}_{M,-,2}$ so that $\Psi_+|_{M_+}$, $\Psi_{-,1}|_{M_{-,1}}$ $\Psi_{-,2}|_{M_{-,2}}$ are identities. Set $g(x) = h(\Psi_+(x))$ for $x \in \overline{\Omega}_{M',+}$ and $g(x) = h(\Psi_{-,\ell}(x))$ for $x \in \overline{\Omega}_{M',-,\ell}$, $\ell = 1, 2$.

Define g as above for all j and for all components M of $h^{-1}(y_j)$ homeomorphic to figure eight. Since by our assumptions the sets Ω_M are disjoint g is well-defined on these components. For those x which do not belong to any of these components we put $g(x) = h(x)$.

Then, g is the desired function. $\square$

Lemma 5.27. *A generic $f \in \mathcal{C}(S^2, I)$ has the property that if $y \in f(S^2)$ and $M \in \mathrm{Comp}(f^{-1}(y))$, then M is the boundary of each component of $S^2 \setminus M$.*

Proof. Let $\mathcal{B}$ be a countable basis for the topology on S^2 consisting of open balls, $k \in \mathbb{N}$ and $V \in \mathcal{B}$. Let $\mathcal{G}_{V,k}$ be the set of those $f \in \mathcal{C}(S^2, I)$ for which there exists $y \in f(S^2)$, a component M of $f^{-1}(y)$ and $x \in M$ such that the following two conditions hold:

- $|f(p) - f(x)| = |f(p) - y| \geqslant \frac{1}{k}$ for all $p \in V$, and
- $d(x, \overline{U}) \geqslant \frac{1}{k}$ where U is the component of $S^2 \setminus M$ with $V \subseteq U$.

It will suffice to show that $\mathcal{G}_{V,k}$ is nowhere dense and closed.

Let us first proceed to show that $\mathcal{G}_{V,k}$ is closed. Let $\{f_n\}$ be a sequence of functions in $\mathcal{G}_{V,k}$ which converges uniformly to some function $f \in \mathcal{C}(S^2, I)$. Let $\{y_n\}$, $\{M_n\}$, $\{x_n\}$, $\{U_n\}$ be such that $y_n \in f_n(S^2)$, $M_n \in \mathrm{Comp}(f_n^{-1}(y_n))$, $x_n \in M_n$, $|f(p) - y_n| \geqslant 1/k$ for all $p \in V$ and U_n is the component of $S^2 \setminus M_n$ containing V and satisfying $d(x_n, \overline{U}_n) \geqslant 1/k$. Without loss of generality, we may assume that $\{y_n\}$ converges to some y, $\{M_n\}$ converges to some M, $\{x_n\}$ converges to some x, and $\{\overline{U}_n\}$ converges in the Hausdorff metric to some set K. We note that M and K are continua, $f(M) = y$, $x \in M$, $d(x, K) \geqslant 1/k$, $|f(p) - y| \geqslant 1/k$ for all $p \in V$ and $V \subseteq K$. Let N be the component of $f^{-1}(y)$ containing M. Denote by U the component of $S^2 \setminus N$ with $V \subseteq U$. We need to show that $U \subseteq K$ to complete the proof of the fact that $\mathcal{G}_{V,k}$ is closed. Let $p \in U$. As $V \subseteq U$, there is an arc $A \subseteq U$ such that $p \in A$ and $A \cap V \neq \emptyset$. For sufficiently large n, we have that $A \cap M_n = \emptyset$; for otherwise, $A \cap M \neq \emptyset$ would imply $A \cap N \neq \emptyset$, contradicting that $A \subseteq U \subseteq S^2 \setminus N$. As A is connected, $A \cap V \neq \emptyset$, and $A \subseteq S^2 \setminus M_n$ for sufficiently large n, we have that $A \subseteq U_n$ for sufficiently large n. As $\{\overline{U}_n\}$ converges to K in the Hausdorff metric, we have that $p \in K$. Hence, $\mathcal{G}_{V,k}$ is closed.

Now the fact that $\mathcal{G}_{V,k}$ is nowhere dense simply follows from Proposition 5.26. □

Corollary 5.28. *A generic $f \in \mathcal{C}(S^2, I)$ has the property that if $y \in f(S^2)$ and $M \in$ Comp$(f^{-1}(y))$ separates S^2 into three pieces, then M is a Lakes of Wada continuum.*

Proof. This simply follows from Theorems 5.17 and 5.27. □

Corollary 5.29. *A generic function $f \in \mathcal{C}(S^2, I)$ has the property that there is a countable dense set $D \subseteq f(S^2)$ such that for all $y \in D$ there is a component M of $f^{-1}(y)$ which is a Lakes of Wada continuum.*

Proof. This follows from Corollary 5.28 and Theorem 5.23. □

5.4. Existence of pseudoarcs and pseudocircles

In this subsection we show that for a generic $f \in \mathcal{C}(S^2, I)$, for almost all $y \in f(S^2)$, all components of $f^{-1}(y)$ are either points, pseudoarcs or pseudocircles. Furthermore, for a generic f, for all $y \in (\min f, \max f)$ there are components of $f^{-1}(y)$ which are pseudoarcs.

Given a set E we denote its $\varepsilon > 0$ neighborhood by $B_\varepsilon(E) = \{x\colon d(x, E) < \varepsilon\}$, the closure of $B_\varepsilon(E)$ is denoted by $\overline{B}_\varepsilon(E)$.

The following lemma is a standard fact from the plane topology.

Lemma 5.30. *Let $M \subseteq S^2$ be a continuum, $U \subseteq S^2$ be a connected open set with $M \subseteq U$ and $\alpha, \beta > 0$. Then, there is an arc $\gamma \subseteq U$ and $0 < \rho < \beta$ such that*

- *$d_H(M, \gamma) < \alpha$,*
- *$\overline{B}_\rho(\gamma) \subseteq U$, and*
- *$\overline{B}_\rho(\gamma)$ and $\overline{B}_{\rho/2}(\gamma)$ admit 4γ and 2γ maps onto γ, respectively.*

Proof. The proof basically follows from the well-known fact that each continuum in S^2 can be approximated by an arc. $\square$

For $A, B \subseteq S^2$ we put $\operatorname{dist}(A, B) = \inf\{d(x, y)\colon\ x \in A,\ y \in B\}$.

Theorem 5.31. *A generic $f \in \mathcal{C}(S^2, I)$ has the property that if*

$$y \in \big(\min\big(f\big(S^2\big)\big), \max\big(f\big(S^2\big)\big)\big),$$

then $f^{-1}(y)$ contains a component which is a pseudoarc.

Proof. In light of Theorem 5.15, it will suffice to show that a generic $f \in \mathcal{C}(S^2, I)$ has the property that for all $y \in (\min f, \max f)$ there is a component of $f^{-1}(y)$ which is arc-like.

Given $f \in C(S^2, I)$, $y \in (\min f, \max f)$ and $\varepsilon, \rho > 0$ we call a subset of S^2 an ε-ρ-y-*worm* for f and denote it by W if

- $W = \overline{B}_\rho(\gamma)$ for some arc γ,
- $f(x) > y + \varepsilon$ for all $x \in \partial W$,
- $f(x) < y - \varepsilon$ for all $x \in \gamma$, and
- there exists a 4ρ-map, $\Gamma_W : W \to \gamma$.

It will suffice to show that for a generic $f \in \mathcal{C}(S^2, I)$ and $y \in (\min f, \max f)$ there exists a sequence of ε_k-ρ_k-y-*worms* $\{W_k\}$ for f such that

- $W_{k+1} \subseteq W_k$,
- $\varepsilon_k \to 0$, $\rho_k \to 0$, and
- $\operatorname{diam}(W_k) \not\to 0$.

Indeed, then it is not difficult to see that $\bigcap_{k=1}^{\infty} W_k$ is an arc-like continuum which is a component of $f^{-1}(y)$.

Let $\phi \in \mathbf{WB}$, $\mu, \nu \in \mathbb{N}$. We construct $\phi^* = \phi^*(\mu, \nu)$ as follows. Partition $\phi(S^2)$ into 2^μ many equal pieces and let J consist of the endpoints of these partition intervals. Let $\mathcal{J}$ be the set of all components of $\phi^{-1}(y)$ such that $y \in J$. Note that since $\phi \in \mathbf{WB}$, each element of $\mathcal{J}$ is either a point, a circle or a figure-eight. Let $0 < \eta < 2^{-\nu}$ be small enough so that if M and M' are two distinct elements of $\mathcal{J}$, then $\overline{B}_\eta(M) \cap \overline{B}_\eta(M') = \emptyset$ and if $d(x, M) < \eta$ for some $M \in \mathcal{J}$, then $|\phi(x) - \phi(M)| < 2^{-\mu}\lambda_1(\phi(S^2))$, where $\lambda_1(\phi(S^2))$ denotes the Lebesgue measure of $\phi(S^2)$.

Fix $M \in \mathcal{J}$ with $\phi(M) = y$. Using Lemma 5.30, choose an arc $\gamma_M \subseteq B_\eta(M)$ and $0 < \rho_M < \eta$ so that

$$\operatorname{diam}(\gamma_M) - \rho_M > \big(1 - (1/2)^\nu\big)\operatorname{diam}(M), \tag{7}$$

$\overline{B}_{\rho_M}(\gamma_M) \subseteq B_\eta(M)$, and $\overline{B}_{\rho_M}(\gamma_M)$ and $\overline{B}_{\rho_M/2}(\gamma_M)$ admit $4\rho_M$ and $2\rho_M$ maps onto γ_M, respectively. Set $V_M = \overline{B}_{\rho_M}(\gamma_M)$ and $W_M = \overline{B}_{\rho_M/2}(\gamma_M)$. Assume that $\widetilde{M} \subseteq W_M$ is a continuum separating ∂W_M from γ_M. For any $x \in \partial W_M$ there exists $x' \in \gamma_M$ and an arc $\gamma' \subseteq W_M$ of length $\rho_M/2$ connecting x and x'. This arc should intersect $\widetilde{M}$. Hence

$$\operatorname{diam}\big(\widetilde{M}\big) \geqslant \operatorname{diam}(\gamma_M) - \rho_M. \tag{8}$$

Now we define ϕ_M^* on V_M as follows. If $x \in \partial V_M$, then put $\phi_M^*(x) = \phi(x)$. If $x \in \gamma_M$, then set $\phi_M^*(x) = y - 3 \cdot 2^{-\mu}\lambda_1(\phi(S^2))$. If $x \in \partial W_M$, then set $\phi_M^*(x) = y + 3 \cdot 2^{-\mu}\lambda_1(\phi(S^2))$. Now using Tietze Extension Theorem, extend ϕ_M^* to all of V_M so that $\phi_M^*(V_M) = [y - 3 \cdot 2^{-\mu}\lambda_1(\phi(S^2)), y + 3 \cdot 2^{-\mu}\lambda_1(\phi(S^2))]$.

We do this for all $M \in \mathcal{J}$ and obtain V_M, W_M and ϕ_M^*. We let $\phi^* = \phi_M^*$ on V_M and $\phi^* = \phi$ otherwise. We put $\delta = 2^{-\mu}\lambda_1(\phi(S^2))$. Let $g \in \mathcal{C}(S^2, I)$ be such that $\|\phi^* - g\| < \delta$. We note that ϕ^*, g satisfy the following properties:

(1) $\|\phi - \phi^*\| \leqslant 4 \cdot 2^{-\mu}\lambda_1(\phi(S^2)) \leqslant 4 \cdot 2^{-\mu}$,
(2) $\|\phi - g\| < 5 \cdot 2^{-\mu}\lambda_1(\phi(S^2)) \leqslant 5 \cdot 2^{-\mu}$,
(3) if $y \in \phi(S^2)$ and M is some component of $\phi^{-1}(y')$ for some $y' \in J$ nearest to y, then W_M is a $2^{-\mu}\lambda_1(\phi(S^2))$-$\rho_M$-$y$ worm for ϕ^* with respect to some arc γ, and
(4) property (3) holds when ϕ^* is replaced by g.

Now we proceed to construct our desired dense G_δ set. Choose $\{f_m \in \mathbf{WB}\colon m = 1, 2, \ldots\}$ dense in $C(S^2, I)$. For each positive integer $n \in \mathbb{N}$, obtain $h_{m,n} = f_m^*(m+n, n) = \phi^*(\mu, \nu)$ from $f_m = \phi$ in the above fashion using $\mu = m + n$ and $\nu = n$. Let $\delta_{m,n} = 2^{-(m+n)}\lambda_1(f_m(S^2))$.

Our desired dense G_δ is $\mathcal{G} = \bigcap_{n=1}^\infty \bigcup_{m=1}^\infty B_{\delta_{m,n}}(h_{m,n})$.

Now we fix a $g \in \mathcal{G}$ and a $y \in (\min g, \max g)$. Let $d = \min\{y - \min g, \max g - y\}$. Let $N \in \mathbb{N}$ be such that $2^{-N} < d/5$. By property (2) above, we have that for all $n > N$ and $m \in \mathbb{N}$, if $g \in B_{\delta_{m,n}}(h_{m,n})$, then $y \in [\min f_m, \max f_m]$.

Now we choose $n_1 > N$ and $m_1 \in \mathbb{N}$ such that $g \in B_{\delta_{m_1,n_1}}(h_{m_1,n_1})$. Using the fact that $y \in f_{m_1}(S^2)$ and property (4) above, there is an ε_1-ρ_1-y worm, W_1 for g with respect to some arc γ_1 and some $0 < \varepsilon_1 \leqslant 2^{-(m_1+n_1)}$, $0 < \rho_1 < 2^{-n_1}$ such that (7) and (8) hold.

Now suppose that $k > 1$, $\{W_i\}_{i=1}^k$, $\{\gamma_i\}_{i=1}^k$, $\{n_i\}_{i=1}^k$, $\{m_i\}_{i=1}^k$, $\{\varepsilon_i\}_{i=1}^k$, and $\{\rho_i\}_{i=1}^k$ have been constructed so that

(a) $\{n_i\}_{i=1}^k$ is an increasing sequence,
(b) $\varepsilon_i \leqslant 2^{-(m_i+n_i)}$ for all $1 \leqslant i \leqslant k$,
(c) $\rho_i < 2^{-n_i}$ for all $1 \leqslant i \leqslant k$,
(d) $W_{i+1} \subseteq W_i$ for all $1 \leqslant i < k$,
(e) for all $1 \leqslant i \leqslant k$, W_i is an ε_i-ρ_i-y worm for g with respect to an arc γ_i, and
(f) for all $1 < i \leqslant k$, $\operatorname{diam}(\gamma_i) - \rho_i > (1 - (1/2)^i)(\operatorname{diam}(\gamma_{i-1}) - \rho_{i-1}) > 0$, and hence $\operatorname{diam}(\gamma_i) - \rho_i > \prod_{j=2}^i (1 - (1/2)^j)(\operatorname{diam}(\gamma_1) - \rho_1) > \prod_{j=2}^\infty (1 - (1/2)^j)(\operatorname{diam}(\gamma_1) - \rho_1) > 0$.

For each $n > n_k$, let $m(n)$ be chosen so that $g \in B_{\delta_{m(n),n}}(h_{m(n),n})$. Then $\{h_{m(n),n}\}$ and, by property (2), $\{f_{m(n)}\}$ converge uniformly to g. Using the fact that $g(\partial(W_k)) \subseteq (y + \varepsilon_k, 1]$ and $g(\gamma_k) \subseteq [0, y - \varepsilon_k)$, we have that for sufficiently large n, $f_{m(n)}(\partial W_k) \subseteq (y + \varepsilon_k, 1]$ and $f_{m(n)}(\gamma_k) \subseteq [0, y - \varepsilon_k)$. Again, using the fact that $\{f_{m(n)}\}$ converges uniformly to g, we have that $f_{m(n)}^{-1}(y)$ converges in the Hausdorff metric to some subset of $g^{-1}(y)$. If $\eta > 0$ is such that $\operatorname{dist}(g^{-1}(y) \cap W_k, \partial W_k) > \eta/4$ and $\operatorname{dist}(g^{-1}(y) \cap W_k, \gamma_k) > \eta/4$, then for sufficiently large n, we have that $\operatorname{dist}(f_{m(n)}^{-1}(y) \cap W_k, \partial W_k) > \eta/4$ and $\operatorname{dist}(f_{m(n)}^{-1}(y) \cap$

$W_k, \gamma_k) > \eta/4$. Since the mesh of the partition of $f_{m(n)}(S^2)$ is going to zero as $n \to \infty$, we have the following property for sufficiently large n: If y' is an endpoint of the partition interval of $f_{m(n)}(S^2)$ closest to y and M is the component of $f_{m(n)}^{-1}(y')$ contained in W_k and separating γ_k from ∂W_k, then $B_{2^{-n}}(M) \subseteq W_k$ and (8) can be applied to M and γ_k. Choose an $n_{k+1} > n_k$ for which the previous property holds. Set $m_{k+1} = m(n_{k+1})$. Now, let W_{k+1} be the worm associated with M in properties (3) and (4) applied with $\phi = f_{m_{k+1}}$, $\phi^* = f^*_{m_{k+1}} = h_{m_{k+1},n_{k+1}}$ and $g = g$. We let γ_{k+1} be the arc associated with W_{k+1} and $\varepsilon_{k+1}, \rho_{k+1}$ be parameters associated with W_{k+1} and γ_{k+1}. It is clear that properties (a)–(e) of the induction hypothesis are satisfied. To see property (f), we use that $n_1 > 1$, $\{n_i\}$ is increasing, M separates γ_k from ∂W_k, (7) and (8) can be applied for γ_{k+1}, M and γ_k. This implies

$$\begin{aligned}\operatorname{diam}(\gamma_{k+1}) - \rho_{k+1} &> \bigl(1 - (1/2)^{n_{k+1}}\bigr) \cdot \operatorname{diam}(M)\\ &> \bigl(1 - (1/2)^{k+1}\bigr) \cdot \bigl(\operatorname{diam}(\gamma_k) - \rho_k\bigr).\end{aligned}$$

Hence, we have constructed our desired sequence $\{W_k\}$, $\{\gamma_k\}$, $\{n_k\}$, $\{m_k\}$, $\{\varepsilon_k\}$ and $\{\rho_k\}$. □

Lemma 5.32. *Assume $f \in \mathbf{WB}$, $y \in f(S^2)$ and M is a component of $f^{-1}(y)$ homeomorphic to a circle. Given $\varepsilon > 0$ there exists $\eta \in (0, \varepsilon)$ and an ε-mapping Δ_M from $B_\eta(M)$ onto M.*

Proof. This follows from Lemma 4.5. □

Theorem 5.33. *All nondegenerate components of almost all fibers of a generic $f \in \mathcal{C}(S^2, I)$ are arc-like, or circle-like.*

Proof. Assume $f_m \in \mathbf{WB}$ is dense in $C(S^2, I)$.

Fix $m, n > 0$. First choose $y_1, \ldots, y_k$ such that if $y \in f_m(S^2) \setminus \{y_1, \ldots, y_k\}$ then all components of $f_m^{-1}(y)$ are homeomorphic to a circle. Choose $\varepsilon_{m,n} > 0$ so that $4k\varepsilon_{m,n} < 2^{-n}$. Put $E_{m,n} = \bigcup_{\ell=1}^k (y_\ell - 2\varepsilon_{m,n}, y_\ell + 2\varepsilon_{m,n})$ and $E'_{m,n} = \bigcup_{\ell=1}^k (y_\ell - \varepsilon_{m,n}, y_\ell + \varepsilon_{m,n})$. Then, $\lambda_1(E_{m,n}) < 2^{-n}$.

Now fix $y \in f_m(S^2) \setminus \{y_1, \ldots, y_k\}$. By using Lemma 5.32 with $\varepsilon = 1/n$ for each component M of $f_m^{-1}(y)$ choose $\eta \in (0, 1/n)$ and a $1/n$-mapping Δ_M from $B_\eta(M)$ onto M. By taking minimum, we can assume that the same η works for all components of $f_m^{-1}(y)$ and if M, M' are different components of $f_m^{-1}(y)$ then $\overline{B}_\eta(M) \cap \overline{B}_\eta(M') = \emptyset$ and $f_m(\overline{B}_\eta(M)) \cap \{y_1, \ldots, y_k\} = \emptyset$. Choose $\rho_y \in (0, \varepsilon_{m,n})$ such that if M is a component of $f_m^{-1}(y)$ then $|f_m(x) - y| > 3\rho_y$ for all $x \in \partial B_\eta(M)$. Let $G_y = f_m^{-1}((y - \rho_y, y + \rho_y)) \cap B_\eta(f_m^{-1}(y))$.

Do the above process for all $y \in f_m(S^2) \setminus \{y_1, \ldots, y_k\}$. The sets $f_m^{-1}((y_\ell - \varepsilon_{m,n}, y_\ell + \varepsilon_{m,n}))$ and the sets G_z for $z \in f_m(S^2) \setminus \{y_1, \ldots, y_k\}$ form an open cover of S^2. Hence, there is a finite cover consisting of sets G_{z_ℓ}, $\ell = 1, \ldots, t$ and of type $f_m^{-1}((y_\ell - \varepsilon_{m,n}, y_\ell + \varepsilon_{m,n}))$, $\ell = 1, \ldots, k$. Denote $\rho_{m,n} = \min\{\rho_{z_\ell}\colon \ell = 1, \ldots, t\}$.

Let $\mathcal{G}_n = \bigcup_{m=1}^\infty B_{\rho_{m,n}}(f_m)$ and $\mathcal{G} = \bigcap_{n=1}^\infty \mathcal{G}_n$. Clearly, $\mathcal{G}$ is a dense G_δ set in $C(S^2, I)$.

Let $f \in \mathcal{G}$. For each n choose m_n such that $f \in B_{\rho_{m_n,n}}(f_{m_n})$. Let $E_f = \bigcup_{K=1}^\infty \bigcap_{n=K}^\infty E_{m_n,n}$. Then, clearly $\lambda_1(E_f) = 0$. We will show that for all $y \in (\min f,$

$\max f) \setminus E_f$ all nondegenerate components of $f^{-1}(y)$ are either arc-like or circle like. To this end, let M be a nondegenerate component of $f^{-1}(y)$ and $\varepsilon > 0$. Since $y \notin E_f$, there are infinitely many n's such that $y \notin E_{m_n,n}$. Hence, there is n such that $1/n < \varepsilon$ and $y \notin E_{m_n,n}$. Let $x \in M$ and $y' = f_{m_n}(x)$. Since $f \in B_{\rho_{m_n,n}}(f_{m_n})$, $|y' - y| < \rho_{m_n,n} < \varepsilon_{m_n,n}$. Hence, $y' \notin E'_{m_n,n}$. Using this fact, we may obtain z such that $|y' - z| < \rho_z$ and a component M'' of $f_{m_n}^{-1}(z)$ such that $x \in B_{\eta_z}(M'')$. (Constants ρ_z and η_z are associated with the function f_{m_n}.) Now we have that for all $t \in M$,

$$\begin{aligned} \left|f_{m_n}(t) - z\right| &= \left|f_{m_n}(t) - f(t)\right| + \left|f(x) - z\right| \\ &\leqslant \rho_{m_n,n} + \left|f(x) - f_{m_n}(x)\right| + \left|f_{m_n}(x) - z\right| \\ &\leqslant \rho_{m_n,n} + \rho_{m_n,n} + |y' - z| \\ &\leqslant 2\rho_z + \rho_z \\ &= 3\rho_z. \end{aligned}$$

Since for all $t \in \partial B_{\eta_z}(M'')$ we have $|f_{m_n}(t) - z| > 3\rho_z$ there is no point of M on $\partial B_{\eta_z}(M'')$. Therefore, $M \subseteq B_{\eta_z}(M'')$. Hence, $\Delta_{M''}$, restricted to M, is a $1/n$-map of M into M''. Hence, we have constructed an ε-map from M onto an arc or a circle. □

Corollary 5.34. *A generic $f \in \mathcal{C}(S^2, I)$ has the property that for almost all $y \in (\min f, \max f)$, all components of $f^{-1}(y)$ are either points, pseudoarcs or pseudocircles.*

Proof. This follows from Theorems 5.33 and 5.15. □

Theorem 5.35. *A generic function $f \in \mathcal{C}(S^2, I)$ has the property that almost all of its fibers contain pseudocircles as components.*

Proof. This follows from Corollary 5.34, the fact that $f^{-1}(y)$ separates S^2 for any $f \in \mathcal{C}(S^2, I)$ and $y \in (\min f, \max f)$, and the fact that no pseudoarc separates S^2. □

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